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# ON A LOCAL ENERGY DECAY OF SOLUTIONS FOR THE EQUATIONS OF MOTION OF COMPRESSIBLE VISCOUS AND HEAT-CONDUCTIVE GASES IN AN EXTERIOR DOMAIN IN $R^3$

By

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**Abstract.** We consider the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in  $R^3$ . We prove the local energy decay of solutions to the linearized evolution problem in  $L_p$  framework.

## §0. Introduction

Let  $\Omega$  be an exterior domain in  $R^3$  with compact smooth boundary  $\partial\Omega$ . The motion of a compressible viscous and heat-conductive fluid is described by the following system

$$\begin{aligned}
 (0.1) \quad & \rho_t + (v \cdot \nabla)\rho + \rho \cdot \operatorname{div} v = 0 && \text{in } [0, \infty) \times \Omega, \\
 & v_t + (v \cdot \nabla)v = \frac{\mu}{\rho} \cdot \Delta v + \frac{\mu + \mu'}{\rho} \cdot \nabla(\operatorname{div} v) - \frac{\nabla P(\rho, \theta)}{\rho} && \text{in } [0, \infty) \times \Omega, \\
 & \theta_t + (v \cdot \nabla)\theta + \frac{\theta \cdot \partial_\theta P}{\rho \cdot c} \cdot \operatorname{div} v = \frac{k}{\rho \cdot c} \Delta \theta + \frac{\Psi}{\rho \cdot c} && \text{in } [0, \infty) \times \Omega, \\
 & v|_{\partial\Omega} = v|_\infty = 0, \quad \theta|_{\partial\Omega} = \theta|_\infty = \bar{\theta} && \text{on } [0, \infty) \times \partial\Omega, \\
 & (\rho, v, \theta)(0, x) = (\rho_0, v_0, \theta_0)(x) && \text{in } \Omega,
 \end{aligned}$$

where  $\rho$  is the density,  $v = (v_1, v_2, v_3)$  the velocity,  $\theta$  the absolute temperature,  $P = P(\rho, \theta)$  the pressure,  $\mu$  and  $\mu'$  the viscosity coefficients,  $k$  the coefficient of the heat conduction,  $c$  the heat capacity at constant volume and  $\Psi$  is the dissipation

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function:

$$\Psi = \frac{\mu}{2}(\partial_k v_j + \partial_j v_k)^2 + \mu'(\partial_j v_j)^2.$$

The existence theorems of unique solution local in time for the system (0.1) were obtained by Nash [15], Itaya [7,8] for the initial value problem, and by Tani [22] for the initial boundary value problem. On the other hand the existence theorem of global solution in time for the system (0.1) were obtained by Matsumura and Nishida [12,13], Ponce [17] for the initial value problem, and by Matsumura and Nishida [14] for the initial boundary value problem in  $L_2$ -framework for sufficiently small initial data. Also Ströhmer [21] proved the global in time existence theorem for small initial data in a bounded domain in  $L_q$ -framework. In particular, Matsumura and Nishida [14] showed that this solution approaches the stationary state as  $t \rightarrow \infty$ , and also Deckelnick [3,4] gave some estimates for the decay rate in an exterior domain. But this decay rate is weaker than that of Matsumura and Nishida [12] and Ponce [17] in Cauchy problem.

In this paper, we shall give the local energy decay of solutions for the linearized equations of nonlinear problem (0.1). Although this system has a hyperbolic part that is the density  $\rho$ , these solutions have the same decay rate as well-known results of the local energy decay of some parabolic equations, for example Stokes operator and Oseen operator. (cf. Iwashita [9], Kobayashi and Shibata [11], Iwashita and Shibata [10] and Shibata [18].) In particular, this decay rate corresponds to that of Matsumura and Nishida [12] and Ponce [17].

Now, we introduce the linearized equations for the system (0.1) below.

$$\begin{aligned} \rho_t + \gamma \operatorname{div} v &= f_1 && \text{in } [0, \infty) \times \Omega, \\ v_t - \alpha \Delta v - \beta \nabla(\operatorname{div} v) + \gamma \nabla \rho + \omega \nabla \theta &= f_2 && \text{in } [0, \infty) \times \Omega, \\ \theta_t - \kappa \Delta \theta + \omega \operatorname{div} v &= f_3 && \text{in } [0, \infty) \times \Omega, \\ v|_{\partial\Omega} = v|_{\infty} = 0, \theta|_{\partial\Omega} = \theta|_{\infty} = 0 &&& \text{on } [0, \infty) \times \partial\Omega, \\ (\rho, v, \theta)(0, x) &= (\rho_0, v_0, \theta_0)(x) && \text{in } \Omega, \end{aligned} \tag{0.2}$$

where  $\alpha, \gamma, \kappa, \omega$  are positive numbers and  $\beta$  is a nonnegative number.

System (0.2) was given by Matsumura and Nishida [12] and Ponce [17]. They seek solutions for the system (0.1) in a neighborhood of a constant state  $(\rho, v, \theta) = (\bar{\rho}_0, 0, \bar{\theta}_0)$  where  $\bar{\rho}_0, \bar{\theta}_0$  are positive constants under the following assumptions:

- (1)  $\mu, \mu'$  are constants  $\mu > 0$  and  $\frac{2}{3}\mu + \mu' \geq 0$ .  
 (2)  $c, k$  are positive constants.  
 (3)  $P$  is a known function of  $\rho, \theta$ , smooth in a neighborhood of  $(\bar{\rho}_0, \bar{\theta}_0)$   
 where  $\frac{\partial P}{\partial \rho}, \frac{\partial P}{\partial \theta} > 0$ .

Note that the assumption (1) is stronger than ours because they also study the Neumann boundary condition.

In equations (0.1), put  $\alpha = (\mu/\bar{\rho}_0)$ ,  $\beta = (\mu + \mu')/\bar{\rho}_0$ ,  $\gamma = \{(\partial P/\partial \rho)(\bar{\rho}_0, \bar{\theta}_0)\}^{1/2}$ ,  $\kappa = (k/c\bar{\rho}_0)$  and put  $\omega = (1/\bar{\rho}_0) \cdot (\partial P/\partial \theta)(\bar{\rho}_0, \bar{\theta}_0)\{\bar{\theta}_0/c\}^{1/2}$ . Then using the notation  $(\rho, v, \theta)$  for the vector  $(1/\bar{\rho}_0)\{(\partial P/\partial \rho)(\bar{\rho}_0, \bar{\theta}_0)\}^{1/2}\rho, v, \{c/\bar{\theta}_0\}^{1/2}\theta)$ , we can obtain the equations (0.2).

Concerning the linearized equations (0.2), Matsumura and Nishida [12] gave the spectral analysis and energy estimates of solutions in  $L_2$ -sense and Ponce [17] the  $L_p - L_q$  estimates for solutions in  $R^3$ , respectively. Ströhmer [20] showed that the operator  $-A$  generates an analytic semigroup in a bounded domain. But the results for the case of an exterior domain were not known. Therefore we shall start with a result for the case of an exterior domain.

Our main results are the following. Let  $1 < q < \infty$ ,  $m$  be an integer and set

$$X_q^m(\Omega) = \{^T u; u \in W_q^{m+1}(\Omega) \times W_q^m(\Omega) \times W_q^m(\Omega)\}, \quad X_q(\Omega) = X_q^0(\Omega),$$

where  $^T u$  means the transposed  $u$ . Define the  $5 \times 5$  matrix operator  $A$  by the relation:

$$(0.3) \quad A = \begin{pmatrix} 0 & \gamma \operatorname{div} & 0 \\ \gamma \nabla & -\alpha \Delta - \beta \nabla \operatorname{div} & \omega \nabla \\ 0 & \omega \operatorname{div} & -\kappa \Delta \end{pmatrix},$$

with the domain:

$$\mathcal{D}(A) = \{^T u; u = \{\rho, v, \theta\} \in W_q^1(\Omega) \times W_q^2(\Omega) \times W_q^2(\Omega), \\ v|_{\partial\Omega} = 0, \theta|_{\partial\Omega} = 0 \text{ on } \partial\Omega\}.$$

Let  $P$  be projection from  $\mathcal{D}(A)$  into  $\{^T \{v, \theta\}; \{v, \theta\} \in W_q^2(\Omega) \times W_q^2(\Omega), v|_{\partial\Omega} = 0, \theta|_{\partial\Omega} = 0 \text{ on } \partial\Omega\}$  and  $\rho(-A)$  be the resolvent set of the operator  $-A$ . Then

**THEOREM A.** *Let  $1 < q < \infty$ . Then  $-A$  is a closed linear operator in  $X_q(\Omega)$  and*

$$(0.4) \quad \rho(-A) \supset \Sigma = \{\lambda \in \mathbb{C}; C \operatorname{Re} \lambda + (\operatorname{Im} \lambda)^2 > 0\},$$

where  $C$  is a constant depending only on  $\alpha, \beta, \gamma, \kappa$  and  $\omega$ . Moreover, the following properties are valid: There exist positive constants  $\lambda_0$  and  $\delta < (\pi/2)$  such that

$$|\lambda| \|(\lambda + A)^{-1} f\|_{X_q(\Omega)} + \|P(\lambda + A)^{-1} f\|_{2,q,\Omega} \leq C(\lambda_0, \delta, m) \|f\|_{X_q(\Omega)}$$

for any  $\lambda - \lambda_0 \in \Sigma_\delta = \{\lambda \in \mathbb{C}; |\arg \lambda| \leq \pi - \delta\}$  and any  $f \in X_q(\Omega)$ .

Theorem A means that  $-A$  generates an analytic semigroup  $e^{-tA}$  on  $X_q(\Omega)$ . Then let  $B_b = \{x \in \mathbb{R}^3; |x| < b\}$ ,  $\Omega_b = \Omega \cap B_b$  and setting

$$(0.5) \quad Y_{q,b}(\Omega) = \left\{ u = {}^T\{\rho, v, \theta\} \in X_q(\Omega); \quad u(x) = 0 \text{ for } x \in \mathbb{R}^3 \setminus B_b, \int_{\Omega_b} \rho(x) dx = 0 \right\},$$

we have

**THEOREM B** (local energy decay). *Let  $1 < q < \infty$  and let  $b_0$  be a fixed number such that  $B_{b_0} \supset \mathbb{R}^3 \setminus \Omega$ . Suppose that  $b > b_0, u = {}^T\{\rho, v, \theta\} \in Y_{q,b}(\Omega)$ . Then the following estimates are valid: for  $M \geq 0$  integer,  $u \in Y_{q,b}(\Omega)$  and  $t > 0$*

$$(0.6) \quad \|\partial_t^M e^{-tA} u\|_{X_q(\Omega_b)} + \|\partial_t^M e^{-tA} u\|_{2,q,\Omega_b} \leq C(q, b, M) t^{-3/2-M} \|u\|_{X_q(\Omega)}.$$

**REMARK.** In dealing with the system (0.2), it is natural to introduce the base space  $X_q(\Omega)$  without the condition  $\int_\Omega \rho(x) dx = 0$  because the Stokes formula does not hold in an exterior domain. Hence we shall treat the case  $\int_\Omega \rho(x) dx \neq 0$  also. In this case, roughly speaking, since  $\lambda = 0$  seems to be a pole in the sense of §1 (1.22), it is difficult to expect the same results in Theorem B. Therefore, we decompose the semigroup  $e^{-tA}$  as the following and by using Theorem B we have

**COROLLARY C.** *Let*

$$(0.7) \quad X_{q,b}(\Omega) = \{u \in X_q(\Omega); u(x) = 0 \text{ for } x \in \mathbb{R}^3 \setminus B_b\}.$$

Taking  $\varphi \in C_0^\infty(\Omega_b)$  so that  $\int_{\Omega_b} \varphi(x) dx = 1$ , for  $u = {}^T\{\rho, v, \theta\} \in X_{q,b}(\Omega)$ , we have the following representation

$$(0.8) \quad e^{-tA} u = T_1(b, \varphi, t)u + T_2(b, \varphi, t)u$$

where  $e_j$  ( $j = 1, 2, \dots, 5$ ) are unit row vectors in  $R^5$ ,  $N_D \mathbf{u} = \int_D \rho(x) dx$  and

$$\begin{aligned} T_1(b, \varphi, t) \mathbf{u} &= e^{-tA} \{ \mathbf{u} - (N_{\Omega_b} \mathbf{u}) \cdot \varphi \mathbf{e}_1 \}, \\ T_2(b, \varphi, t) \mathbf{u} &= (N_{\Omega_b} \mathbf{u}) \left\{ \varphi \cdot \mathbf{e}_1 - \gamma \int_0^t e^{-sA} \begin{pmatrix} 0 \\ \nabla \varphi \\ 0 \end{pmatrix} ds \right\}. \end{aligned}$$

Moreover, the following estimates are valid: for  $M \geq 0$  integer,  $\mathbf{u} \in X_{q,b}(\Omega)$  and  $t > 0$

$$(0.9) \quad \begin{aligned} &\| \partial_t^M T_1(b, \varphi, t) \mathbf{u} \|_{X_q(\Omega_b)} + \| \partial_t^M \mathbf{P} T_1(b, \varphi, t) \mathbf{u} \|_{2,q,\Omega_b} \\ &\leq C(q, b, \varphi, M) t^{-3/2-M} \| \mathbf{u} \|_{X_q(\Omega)}, \end{aligned}$$

$$(0.10) \quad \begin{aligned} &\| \partial_t^{M+1} T_2(b, \varphi, t) \mathbf{u} \|_{X_q(\Omega_b)} + \| \partial_t^{M+1} \mathbf{P} T_2(b, \varphi, t) \mathbf{u} \|_{2,q,\Omega_b} \\ &\leq C(q, b, \varphi, M) t^{-3/2-M} \| \mathbf{u} \|_{X_q(\Omega)}. \end{aligned}$$

The most important part of the proof of our main results is the cutoff technique in Shibata [18]. In §1, the same resolvent estimates of the operator  $-A$  in a bounded domain as in Ströhmer [20] are proved. The difference between ours and Ströhmer [20] are the following:

(i) We shall show that the resolvent set of the operator  $-A$  contains a parabolic region,

(ii) We do not assume that  $\int_{\Omega} \rho(x) dx = 0$ . (see Remark.)

The regularity of resolvent  $(\lambda + A)^{-1}$  in  $R^3$  near  $\lambda = 0$  is investigated in §2, which is the essential point of our proof of Theorem B. The proof of Theorem A in §3 and a construction of a parametrix of the exterior stationary problem in §4 are done by the method of cutoff technique. And then, with the help of a theorem concerning the relationship between the regularity of functions and the decay rate of their Fourier image, which was given by Shibata [18], we prove Theorem B in §5. Since the resolvent set contains a parabolic region, we can not take the same path of integration for the Laplace transform between the resolvent and semi-group as in Iwashita [9] etc. Hence we shall use the same way as in Kobayashi and Shibata [11].

Notations. Three dimensional row vector valued functions are denoted with bold-face letter, for example  $\mathbf{u} = (u_1, u_2, u_3)$ . As usual, we put

$$\begin{aligned} \partial_t &= \partial / \partial t; \quad \partial_j = \partial / \partial x_j; \quad \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2; \\ \partial_x^\alpha &= \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3), \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3; \end{aligned}$$

$$\partial_x^m p = (\partial_x^\alpha p; |\alpha| = m); \quad \bar{\partial}_x^m p = (\partial_x^\alpha p; |\alpha| \leq m);$$

$$\operatorname{div} \mathbf{u} = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3;$$

Sobolev spaces of vector valued functions are used, as well as of scalar valued functions. Thus, if  $D$  is any domain in  $R^3$ , we put

$$\|u\|_{q,D} = \left( \int_D |u(x)|^q dx \right)^{1/q}; \quad \|\mathbf{u}\|_{q,D} = \left( \sum_{j=1}^3 \|u_j\|_{q,D}^q \right)^{1/q};$$

$$\|u\|_{m,q,D} = \|\bar{\partial}_x^m u\|_{q,D}; \quad \|\mathbf{u}\|_{m,q,D} = \|\bar{\partial}_x^m \mathbf{u}\|_{q,D}; \quad (\mathbf{u}, \mathbf{v}) = \int_D \mathbf{u}(x) \cdot \overline{\mathbf{v}(x)} dx.$$

$L_q(D)$  denotes the usual  $L_q$  space on  $D$ ,  $W_q^m(D) = \{u \in L_q(D); \|u\|_{m,q,D} < \infty\}$ ,  $\mathcal{S}'$  the set of all tempered distributions on  $R^3$  and  $C_0^\infty(D)$  the set of all functions of  $C^\infty(R^3)$  whose support is contained in  $D$ . For function spaces of three dimensional vector valued functions, we use the bold letters, that is for example,  $L_q(D) = \{L_q(D)\}^3$  likewise for  $W_q^m(D)$ . To denote various constants, we use the same letter  $C$  and  $C(A, B, \dots)$  means that the constant depends on the qualities  $A, B, \dots$ . For two Banach spaces  $X$  and  $Y$ ,  $\mathcal{B}(X, Y)$  denotes the set of all bounded linear operators from  $X$  into  $Y$  and  $\|\cdot\|_{\mathcal{B}(X,Y)}$  means its operator norm.

### §1. Stationary problem in a bounded domain

In this section we consider the stationary problem in a bounded domain  $D$  in  $R^3$  with smooth boundary  $\partial D$ ;

$$(1.1a) \quad \lambda \rho + \gamma \cdot \operatorname{div} \mathbf{v} = f_1 \quad \text{in } D,$$

$$(1.1b) \quad \lambda \mathbf{v} - \alpha \Delta \mathbf{v} - \beta \nabla(\operatorname{div} \mathbf{v}) + \gamma \cdot \nabla \rho + \omega \cdot \nabla \theta = f_2 \quad \text{in } D,$$

$$(1.1c) \quad \lambda \theta - \kappa \Delta \theta + \omega \cdot \operatorname{div} \mathbf{v} = f_3 \quad \text{in } D,$$

$$(1.1d) \quad \mathbf{v}|_{\partial D} = 0 \quad \text{on } \partial D,$$

$$(1.1e) \quad \theta|_{\partial D} = 0 \quad \text{on } \partial D.$$

here  $\lambda$  is a complex parameter.

We shall prepare some results to show a unique existence of solutions to (1.1). The following proposition is concerned the existence theorem of solutions to the Stokes equations.

**PROPOSITION 1.1 ([2]).** *Let  $1 < q < \infty$ ,  $m$  be an integer  $\geq 0$  and let  $D \subset R^3$  be a bounded domain with smooth boundary  $\partial D$ . Then for every  $\mathbf{f} \in W_q^m(D)$  and every  $g \in W_q^{m+1}(D)$  with  $\int_D g(x) dx = 0$  there exists a unique  $\mathbf{u} \in W_q^{m+2}(D)$  which*

together with some  $p \in W_q^{m+1}(D)$  satisfies

$$-\Delta u + \nabla p = f, \operatorname{div} u = g \text{ in } D,$$

$$u = 0 \text{ on } \partial D.$$

Here  $p$  is unique up to an additive constant. Furthermore, the following estimate is valid:

$$\|u\|_{m+2,q,D} + \|\nabla p\|_{m,q,D} \leq C\{\|f\|_{m,q,D} + \|g\|_{m+1,q,D}\},$$

where  $C = C(D, q, \varepsilon)$  is a constant.

The following proposition is well-known as a general Poincaré's inequality.

**PROPOSITION 1.2** (cf., eg. [5]). *Let  $1 \leq q < \infty$ . There exists a constant  $C > 0$  such that the inequality*

$$\|u\|_{q,D} \leq C\left\{\|\nabla u\|_{q,D} + \left|\int_D u(x) dx\right|\right\},$$

*holds for any  $u \in W_q^1(D)$ . Furthermore, if  $q \neq 1$ ,  $D$  is bounded and if  $u \in W_q^1(D)$  with  $u = 0$  on  $\partial D$ , then we have*

$$\|u\|_{q,D} \leq C\|\nabla u\|_{q,D}.$$

The next result is well-known as the system of Laplacian with Dirichlet boundary conditions.

**PROPOSITION 1.3.** *Let  $1 < q < \infty$  and let  $D \subset R^3$  be a bounded domain (or exterior domain) with smooth boundary  $\partial D$ . Let  $0 < \delta < (\pi/2)$  and  $\kappa > 0$ . Then for every  $\lambda \in \Sigma_\delta$ , every  $f \in L_q(D)$  there exists a unique solution  $u \in W_q^2(D)$  such that*

$$\lambda u - \kappa \Delta u = f \text{ in } D, \quad u = 0 \text{ on } \partial D.$$

Furthermore, the following estimate is valid:

$$|\lambda| \|u\|_{q,D} + \|u\|_{2,q,D} \leq C\|f\|_{q,D}, \quad \|u\|_{3,q,D} \leq C(\lambda)\{\|f\|_{1,q,D} + \|u\|_{q,D}\},$$

where  $C = C(D, q, \delta)$  is a constant.

The following proposition is concerned the existence theorem of solutions to the elastic equations.



PROPOSITION 1.4. Let  $1 < q < \infty$  and let  $D \subset R^3$  be a bounded domain (on exterior domain) with smooth boundary  $\partial D$ . Let  $\alpha$  be a positive number,  $\eta$  be a complex number such that  $\operatorname{Re}\{\alpha + \eta\} > 0$ . Then there exist positive numbers  $\lambda_0$  and  $\delta < (\pi/2)$  satisfying the following conditions: For every  $\lambda - \lambda_0 \in \sum_\delta$ , every  $f \in L_q(D)$  there exists a unique  $u \in W_q^2(D)$  such that

$$(1.2) \quad \lambda u - \alpha \Delta u - \eta \nabla \operatorname{div} u = f \text{ in } D, \quad u|_{\partial D} = 0 \text{ on } \partial D.$$

Furthermore the following estimates is valid:

$$(1.3) \quad |\lambda| \|u\|_{q,D} + \|u\|_{2,q,D} \leq C \|f\|_{q,D}, \quad \|u\|_{3,q,D} \leq C(\lambda) \{\|f\|_{1,q,D} + \|u\|_{q,D}\},$$

where  $C = C(D, q, \delta, \lambda_0, \alpha, \eta)$  is a constant.

PROOF. Since

$$(1.4) \quad \det \begin{pmatrix} -\alpha|\xi|^2 - \eta\xi_1^2 & -\eta\xi_1\xi_2 & -\eta\xi_1\xi_3 \\ -\eta\xi_1\xi_2 & -\alpha|\xi|^2 - \eta\xi_2^2 & -\eta\xi_2\xi_3 \\ -\eta\xi_1\xi_3 & -\eta\xi_2\xi_3 & -\alpha|\xi|^2 - \eta\xi_3^2 \end{pmatrix} = -(\alpha + \eta)\alpha^2|\xi|^6,$$

(1.2) is the elliptic when  $\operatorname{Re}(\alpha + \eta) > 0$ , which means that a priori estimate:

$$|\lambda| \|u\|_{q,D} + \|u\|_{2,q,D} \leq C \{\|f\|_{q,D} + \|u\|_{q,D}\}, \quad \|u\|_{3,q,D} \leq C(\lambda) \{\|f\|_{1,q,D} + \|u\|_{q,D}\},$$

is valid for  $\lambda - \lambda_0 \in \sum_\delta$ . Taking sufficiently large number  $\lambda_0$ , we have (1.3). Define the operator  $T(\lambda; \eta)$  by the relation:

$$(1.5) \quad T(\lambda; \eta)u = \lambda u - \alpha \Delta u - \eta \nabla \operatorname{div} u,$$

with the domain:  $\mathcal{D}(T(\lambda; \eta)) = \{u \in W_q^2(D); u|_{\partial D} = 0\}$ . Then, by (1.3)  $T(\lambda; \eta)$  is densely defined closed operator in  $L_q(D)$  and the range of  $T(\lambda; \eta)$  is closed in  $L_q(D)$ . Since the dual operator of  $T(\lambda; \eta)$  in  $L_q(D)$  is  $T(\bar{\lambda}; \bar{\eta})$  in  $L_p(D)$  where  $(1/p) + (1/q) = 1$ , the closed range theorem means that a unique solution for (1.2) exists in  $L_q(D)$ . Combining this with a priori estimate (1.3), the proof is completed.

Now we will lead to the main theorem in this section. Let  $1 < q < \infty, m$  be an integer and let

$$(1.6) \quad Y_q^m(D) = \left\{ {}^T\{f_1, f_2, f_3\} \in X_q^m(D); \int_D f_1(x) dx = 0 \right\}, \quad Y_q(D) = Y_q^0(D).$$

Define the  $5 \times 5$  matrix operator  $A_D$  by the relation:

$$A_D = \begin{pmatrix} 0 & \gamma \operatorname{div} & 0 \\ \gamma \nabla & -\alpha \Delta - \beta \nabla \operatorname{div} & \omega \nabla \\ 0 & \omega \operatorname{div} & -\kappa \Delta \end{pmatrix},$$

with the domain:  $\mathcal{D}(A_D) = Y_q(D) \cap \mathcal{D}(A)$  i.e.  $A_D$  is the maximal restriction to closed subspace  $Y_q(D)$ . Applying this notation to (1.1), we have

$$(\lambda + A_D)u = f$$

where  $u = {}^T\{\rho, v, \theta\}$  and  $f = {}^T\{f_1, f_2, f_3\}$ . Then

**THEOREM 1.5.** *Let  $1 < q < \infty$  and let  $D \subset R^3$  be a bounded domain with smooth boundary  $\partial D$ . Then,  $A_D$  is a closed linear operator in  $Y_q(D)$  and*

$$\rho(-A_D) \supset \{0\} \cup \Sigma'$$

where  $\Sigma' = \{\lambda \in C; 6(\gamma^2 + \omega^2)\operatorname{Re} \lambda + \alpha(\operatorname{Im} \lambda)^2 > 0\}$ . Moreover, the following properties are valid: There exists a number  $0 < \delta < (\pi/2)$  such that

$$(1.7) \quad |\lambda| \|(\lambda + A_D)^{-1}f\|_{Y_q(D)} + \|P(\lambda + A_D)^{-1}f\|_{2,q,D} \leq C(q, \delta, D) \|f\|_{Y_q(D)}$$

for any  $\lambda \in \Sigma_\delta \cup \{0\}$  and any  $f \in Y_q(\Omega)$ .

**PROOF.** We shall prepare the following three lemmas to prove this theorem.

**LEMMA 1.6.** *Let  $1 < q < \infty$ , and  $D \subset R^3$  be a bounded domain or an exterior domain with smooth boundary  $\partial D$ . Let  $A$  be the operators defined in (0.3) with  $\Omega = D$ . Then there exist positive numbers  $\lambda_0$  and  $0 < \delta < (\pi/2)$  such that if  $u \in \mathcal{D}(A)$  satisfies  $(\lambda + A)u = f$  with  $f \in X_q(D)$ , then the following estimate is valid:*

$$|\lambda| \|u\|_{X_q(D)} + \|Pu\|_{2,q,D} \leq C(q, \lambda_0, \delta, D) \|f\|_{X_q(D)}$$

for  $\lambda - \lambda_0 \in \Sigma_\delta$ .

**PROOF OF LEMMA 1.6.** Let  $u = {}^T\{\rho, v, \theta\}$  and let  $f = {}^T\{f_1, f_2, f_3\}$ . Recall that the equation  $(\lambda + A)u = f$  means that the equations (1.1) hold. Applying Propositions 1.3 and 1.4 to the system  $\lambda - \kappa \Delta$  and  $\lambda - \alpha \Delta - \beta \nabla \operatorname{div}$  in (1.1), we

see that there exist positive number  $\lambda_1$  and  $0 < \delta_1 < (\pi/2)$  such that

$$(1.8a) \quad \begin{aligned} & |\lambda| \|\theta\|_{q,D} + |\lambda|^{1/2} \|\theta\|_{1,q,D} + \|\theta\|_{2,q,D} \\ & \leq C\{\|f_3 - \omega \operatorname{div} v\|_{q,D} + \|\theta\|_{q,D}\}, \end{aligned}$$

and

$$(1.8b) \quad \begin{aligned} & |\lambda| \|v\|_{q,D} + |\lambda|^{1/2} \|v\|_{1,q,D} + \|v\|_{2,q,D} \\ & \leq C\{\|f_2 - \gamma \nabla \rho - \omega \nabla \theta\|_{q,D} + \|v\|_{q,D}\}, \end{aligned}$$

hold for  $\lambda - \lambda_1 \in \Sigma_{\delta_1}$  with  $C$  depending only on  $q, \lambda_1$  and  $\delta_1$ . Furthermore it follows from the equations (1.1a) that

$$(1.9) \quad |\lambda| \|\rho\|_{q,D} \leq \gamma \|v\|_{1,q,D} + \|f_1\|_{q,D},$$

and

$$(1.10) \quad |\lambda| \|\nabla \rho\|_{q,D} \leq \gamma \|v\|_{2,q,D} + \|f_1\|_{1,q,D}.$$

Combining (1.8a), (1.8b), (1.9) and (1.10), and taking sufficiently large number  $\lambda_0$ , we have lemma 1.6.

**LEMMA 1.7.** *Let  $1 < q < \infty$ ,  $m$  be an integer  $\geq 0$  and  $D$  be a bounded domain in  $R^3$  with smooth boundary  $\partial D$ . Then,  $(-A_D)^{-1}$  exists. Furthermore, the following estimate is valid:*

$$\|(-A_D)^{-1} f\|_{Y_q^m(D)} + \|P(-A_D)^{-1} f\|_{m+2,q,D} \leq C(q, m, D) \|f\|_{Y_q^m(D)}$$

for  $f \in Y_q^m(D)$ .

**PROOF OF LEMMA 1.7.** Putting  $u = {}^T\{\rho, v, \theta\}$  and  $f = {}^T\{f_1, f_2, f_3\}$ , we consider the system (1.1) with  $\lambda = 0$  in stead of the equation  $A_D u = f$  in Lemma 1.7. Since it follows from (1.1a), (1.1c) and (1.1e) that

$$(1.11) \quad -\kappa \Delta \theta = f_3 - \frac{\omega}{\gamma} f_1 \text{ in } D, \quad \theta|_{\partial D} = 0 \text{ on } \partial D,$$

and since  $D$  is a bounded domain, there exists a unique solution  $\theta \in W_q^{m+2}(D)$  to (1.11) such that

$$(1.12) \quad \|\theta\|_{m+2,q,D} \leq C \left\| f_3 - \frac{\omega}{\gamma} f_1 \right\|_{m,q,D}.$$

We have by (1.1a), (1.1b) and by (1.1d)

$$(1.13) \quad \begin{aligned} -\alpha \Delta v + \nabla(\gamma \rho) &= f_2 + \frac{1}{\gamma} \beta \nabla f_1 - \omega \cdot \nabla \theta \text{ in } D, \\ \operatorname{div} v &= \frac{f_1}{\gamma} \text{ in } D, \quad v|_{\partial D} = 0 \text{ on } \partial D. \end{aligned}$$

Applying Proposition 1.1 to the system (1.13), there exists a unique pair  $(v, \rho) \in W_q^{m+2}(D) \times W_q^{m+1}(D)$  with  $\int_D \rho(x) dx = 0$  satisfying (1.13) such that

$$(1.14) \quad \begin{aligned} &\|v\|_{m+2,q,D} + \|\rho\|_{m+1,q,D} \\ &\leq C \left\{ \left\| f_2 + \frac{1}{\gamma} \beta \nabla f_1 - \omega \cdot \nabla \theta \right\|_{m,q,D} + \left\| \nabla \frac{f_1}{\gamma} \right\|_{m,q,D} \right\}. \end{aligned}$$

Combining (1.12) with (1.14) implies that this lemma holds.

**LEMMA 1.8.** *Let  $1 < q < \infty$ ,  $\lambda \in \Sigma' \cup \{0\}$  and  $D \subset \mathbb{R}^3$  be a bounded domain with smooth boundary  $\partial D$ . Let  $A$  be the operators defined in (0.3) with  $\Omega = D$ . Then*

$$\operatorname{Ker}(\lambda + A) = \{0\},$$

where  $\operatorname{Ker} T$  is the kernel of the operator  $T$ .

**PROOF OF LEMMA 1.8.** Let  $(\lambda + A)u = 0$ ,  $u = {}^T\{\rho, v, \theta\} \in \mathcal{D}(A)$ . Then we have

$$(1.15a) \quad \lambda \rho + \gamma \cdot \operatorname{div} v = 0 \quad \text{in } D,$$

$$(1.15b) \quad \lambda v - \alpha \Delta v - \beta \nabla(\operatorname{div} v) + \gamma \cdot \nabla \rho + \omega \cdot \nabla \theta = 0 \quad \text{in } D,$$

$$(1.15c) \quad \lambda \theta - \kappa \Delta \theta + \omega \cdot \operatorname{div} v = 0 \quad \text{in } D,$$

$$(1.15d) \quad v|_{\partial D} = 0 \quad \text{on } \partial D,$$

$$(1.15d) \quad \theta|_{\partial D} = 0 \quad \text{on } \partial D.$$

We can assume that  $\lambda \neq 0$  by Lemma 1.7. Noting that  $\operatorname{Re}\{\alpha + \beta + (\gamma^2/\lambda)\} > 0$  when  $\alpha > 0$ ,  $\beta \geq 0$  and  $\lambda \in \Sigma'$ , in view of (1.4), since the systems  $-\kappa \Delta$  and  $-\alpha \Delta - (\beta + (\gamma^2/\lambda)) \nabla \operatorname{div}$  with Dirichlet boundary conditions are elliptic, by boot-strap argument, we see that  $\{\rho, v, \theta\} \in W_q^{\ell+1}(D) \times W_q^{\ell+2}(D) \times W_q^{\ell+2}(D)$  for all integers  $\ell \geq 0$ . When  $2 \leq q < \infty$ , since  $D$  is a bounded domain, we see that  $\{\rho, v, \theta\} \in W_2^1(D) \times W_2^2(D) \times W_2^2(D)$ . When  $1 < q < 2$ , by Sobolev's imbedding theorem,

$\{\rho, v, \theta\} \in W_2^1(D) \times W_2^2(D) \times W_2^2(D)$ . Thus, multiplying (1.15b) by  $\bar{v}$ , integrating the resulting relation over  $D$  and using integration by parts, we have by (1.15a)

$$(1.16) \quad \lambda \|v\|_{2,D}^2 + \alpha \|\nabla v\|_{2,D}^2 + \left(\beta + \frac{\gamma^2}{\lambda}\right) \|\operatorname{div} v\|_{2,D}^2 + \omega(\nabla \theta, v) = 0.$$

Similarly, multiplying (1.15c) by  $\bar{\theta}$ , we have

$$(1.17) \quad \lambda \|\theta\|_{2,D}^2 + \kappa \|\nabla \theta\|_{2,D}^2 + \omega(\operatorname{div} v, \theta) = 0.$$

Since  $\operatorname{Re}\{\omega(\operatorname{div} v, \theta)\} = -\operatorname{Re}\{\omega(\nabla \theta, v)\}$  and since  $\operatorname{Im}\{\omega(\operatorname{div} v, \theta)\} = \operatorname{Im}\{\omega(\nabla \theta, v)\}$ , it follows from (1.16), (1.17) and Schwartz's inequality that

$$(1.18) \quad \operatorname{Re} \lambda \cdot (\|v\|_{2,D}^2 + \|\theta\|_{2,D}^2) + \alpha \|\nabla v\|_{2,D}^2 + \kappa \|\nabla \theta\|_{2,D}^2 + \left(\beta + \frac{\operatorname{Re} \lambda \cdot \gamma^2}{|\lambda|^2}\right) \|\operatorname{div} v\|_{2,D}^2 = 0,$$

$$(1.19) \quad \|v\|_{2,D}^2 = \frac{\gamma^2}{|\lambda|^2} \|\operatorname{div} v\|_{2,D}^2 + \|\theta\|_{2,D}^2 \text{ if } \operatorname{Im} \lambda \neq 0,$$

and

$$(1.20) \quad |\operatorname{Im} \lambda| \|\theta\|_{2,D} \leq \omega \|\operatorname{div} v\|_{2,D}.$$

When  $\operatorname{Re} \lambda \geq 0$ , by (1.18) and (1.19) we have  $\theta = 0$ ,  $v = 0$  in  $D$  because  $\theta = 0$ ,  $v = 0$  on  $\partial D$ , which implies  $\rho = 0$  in  $D$  by (1.15a). When  $\operatorname{Re} \lambda < 0$ , since  $\operatorname{Im} \lambda \neq 0$ , it follows from (1.18), (1.19) and (1.20) that

$$\alpha \|\nabla v\|_{2,D}^2 + \kappa \|\nabla \theta\|_{2,D}^2 + \beta \|\operatorname{div} v\|_{2,D}^2 \leq -2\operatorname{Re} \lambda \left\{ \frac{\gamma^2}{|\lambda|^2} + \frac{\omega^2}{|\operatorname{Im} \lambda|^2} \right\} \|\operatorname{div} v\|_{2,D}^2.$$

Noting that  $\|\operatorname{div} v\|_{2,D}^2 \leq 3\|\nabla v\|_{2,D}^2$  and  $6(\gamma^2 + \omega^2)\operatorname{Re} \lambda + \alpha(\operatorname{Im} \lambda)^2 \geq 0$  when  $\lambda \in \Sigma'$ , we have  $\nabla v = 0$  in  $D$ . Combining this with (1.19) and (1.20) implies that  $\theta = 0$ ,  $v = 0$  in  $D$  and that  $\rho = 0$  in  $D$  by (1.15a). This completes the proof of Lemma 1.8.

WE ARE NOW IN THE POSITION TO PROVE THEOREM 1.5. Note that Lemma 1.7 allows us to show the case  $\lambda \neq 0$ . Putting  $u = {}^T\{\rho, v, \theta\}$  and  $f = {}^T\{f_1, f_2, f_3\}$ , we consider the system (1.1) in stead of the equation  $(\lambda + A_D)u = f$ . In view of Proposition 1.3 and 1.4, fixing a complex number  $\lambda_1 \in \sum_\delta + \lambda_0$ , it follows from (1.1) that for  $\lambda \in \Sigma'$

$$(I + P(\lambda))v = T\left(\lambda_1; \beta + \frac{\gamma^2}{\lambda}\right)^{-1} \left[-\frac{\gamma}{\lambda} \nabla f_1 + f_2 - \omega \nabla (\lambda - \kappa \Delta)^{-1} f_3\right],$$

where  $I$  is the identity operator,

$$P(\lambda) = T\left(\lambda_1; \beta + \frac{\gamma^2}{\lambda}\right)^{-1} [(\lambda - \lambda_1) - \omega^2 \nabla(\lambda - \kappa \Delta)^{-1} \operatorname{div}],$$

$$T\left(\lambda_1; \beta + \frac{\gamma^2}{\lambda}\right) = \text{the operator defined in (1.5),}$$

and

$$(\lambda - \kappa \Delta)^{-1} = \text{the resolvent for the system in Proposition 1.3.}$$

By Proposition 1.3 and 1.4  $P(\lambda)$  is a bounded linear operator from  $\{u \in W_q^2(D); u|_{\partial D} = 0\}$  into  $W_q^3(D) \cap \{u \in W_q^2(D); u|_{\partial D} = 0\}$  which is compactly imbedded into  $\{u \in W_q^2(D); u|_{\partial D} = 0\}$  as follows from Rellich's compactness theorem, and hence  $P(\lambda)$  is a compact operator from  $\{u \in W_q^2(D); u|_{\partial D} = 0\}$  into itself. Noting that by Lemma 1.8 we know that  $I + P(\lambda)$  is injective, by Fredholm's alternative theorem we see that  $I + P(\lambda)$  has the bounded inverse. Hence, setting

$$v = (I + P(\lambda))^{-1} T\left(\lambda_1; \beta + \frac{\gamma^2}{\lambda}\right)^{-1} \left[-\frac{\gamma}{\lambda} \nabla f_1 + f_2 - \omega \nabla(\lambda - \kappa \Delta)^{-1} f_3\right],$$

$$\theta = (\lambda - \kappa \Delta)^{-1} [f_3 - \omega \operatorname{div} v], \quad \rho = \frac{1}{\lambda} [f_1 - \gamma \operatorname{div} v],$$

implies that

$$\rho(-A_D) \supset \Sigma' \cup \{0\}.$$

Furthermore, since the resolvent  $(\lambda + A_D)^{-1}$  is analytic in  $\lambda \in \rho(-A_D)$ , Lemma 1.6 and Lemma 1.7 mean that the estimates (1.7) is valid, which reach the desired conclusion.

**REMARK 1.9.** In Theorem 1.5 we assume that  $\int_D f_1 dx = 0$ , which means that  $\int_D \rho dx = 0$  by the equation (1.1a), (1.1d) and by Stokes formula. When  $\int_D f_1 dx \neq 0$ , taking  $\varphi \in C_0^\infty(D)$  such that  $\int_D \varphi(x) dx = 1$  and define the operators  $N_j = N_j(\varphi, D)$  ( $j = 1, 2, 3$ ) from  $X_q(D)$  into itself by the notations:

$$(1.21) \quad \begin{aligned} N_1 f &= f - (N_D f) \cdot \varphi e_1 \\ N_2 f &= -(N_D f) \begin{pmatrix} 0 \\ \nabla \varphi \\ 0 \end{pmatrix} \text{ for } f = {}^T\{f_1, f_2, f_3\} \in X_q(D), \\ N_3 f &= (N_D f) \varphi \cdot e_1 \end{aligned}$$

where  $e_1$  and  $N_D f$  are the same symbols as in Corollary C. Then we can write  $(\lambda + A)^{-1}$  as follows:

$$(1.22) \quad (\lambda + A)^{-1} = (\lambda + A_D)^{-1} N_1 + \frac{\gamma}{\lambda} (\lambda + A_D)^{-1} N_2 + \frac{1}{\lambda} N_3.$$

Combining this and Theorem 1.5, we see that  $-A$  is a closed linear operator in  $X_q(D)$ ,  $\rho(-A) \supset \Sigma'$  and the following properties are valid:

$$\begin{aligned} & |\lambda| \|(\lambda + A)^{-1} f\|_{X_q(D)} + \|P(\lambda + A)^{-1} f\|_{2,q,D} \\ & \leq C(\delta, q, D) \left\{ \|f\|_{X_q(D)} + \frac{1}{|\lambda|} \|f_1\|_{q,D} \right\} \end{aligned}$$

for any  $\lambda \in \Sigma_\delta$  and any  $f \in X_q(D)$ .

## §2. On the stationary problem in $R^3$

In this section, we shall show the basic estimations of solutions to the following stationary linearized equations in  $R^3$  with a complex parameter  $\lambda$ :

$$\begin{aligned} (2.1) \quad & \lambda \rho + \gamma \cdot \operatorname{div} v = f_1, \\ & \lambda v - \alpha \Delta v - \beta \nabla(\operatorname{div} v) + \gamma \cdot \nabla \rho + \omega \cdot \nabla \theta = f_2 \text{ in } R^3, \\ & \lambda \theta - \kappa \Delta \theta + \omega \cdot \operatorname{div} v = f_3. \end{aligned}$$

By taking Fourier transform on (2.1) we obtain

$$[\lambda \cdot I + \hat{A}(\xi)] \hat{u} = \hat{f},$$

where  $I$  is the identity,  $\mathcal{F}(f) = \hat{f}$  stands for the Fourier transforms of  $f$ ,  $u = {}^T(\rho, v, \theta)$ ,  $f = {}^T(f_1, f_2, f_3)$ . Here  $\hat{A}(\xi)$  is  $5 \times 5$  symmetric matrix as follows:

$$\hat{A}(\xi) = \begin{pmatrix} 0 & i\gamma\xi_k & 0 \\ i\gamma\xi_j & \delta_{jk}\alpha|\xi|^2 + \beta\xi_j\xi_k & i\omega\xi_j \\ 0 & i\omega\xi_k & \kappa|\xi|^2 \end{pmatrix}$$

where  $i = \sqrt{-1}$  and  $\delta_{jk} = 0$  when  $k \neq j$  and  $= 1$  when  $k = j$ . Then we have

$$(2.2a) \quad [\lambda \cdot I + \hat{A}(\xi)]^{-1} = \{\det[\lambda \cdot I + \hat{A}(\xi)]\}^{-1} \cdot \tilde{A}(\lambda; \xi),$$

$$(2.2b) \quad \det[\lambda \cdot I + \hat{A}(\xi)] = (\lambda + \alpha|\xi|^2)^2 F(\lambda; |\xi|),$$

where

$$(2.2c) \quad F(\lambda; |\xi|) = \lambda^3 + (\alpha + \beta + \kappa)|\xi|^2 \lambda^2 + [(\alpha + \beta)\kappa|\xi|^2 + \gamma^2 + \omega^2]|\xi|^2 \lambda + \gamma^2 \kappa |\xi|^4,$$

and  $\tilde{A}(\lambda; \xi) = (\tilde{a}_{ij}(\lambda; \xi))$  is the  $5 \times 5$  matrix and the components are

$$\begin{aligned} \tilde{a}_{11} &= (\lambda + \alpha|\xi|^2)^2 \{ \lambda^2 + (\alpha + \beta + \kappa)|\xi|^2 \lambda + [\omega^2 + (\alpha + \beta)\kappa|\xi|^2] \cdot |\xi|^2 \}, \\ \tilde{a}_{15} &= \tilde{a}_{51} = -\gamma\omega(\lambda + \alpha|\xi|^2)^2 |\xi|^2, \\ (2.2d) \quad \tilde{a}_{1,j} &= \tilde{a}_{j,1} = -i\gamma(\lambda + \alpha|\xi|^2)^2 (\lambda + \kappa|\xi|^2) \xi_{j-1} \quad (j = 2, 3, 4), \\ \tilde{a}_{5,j} &= \tilde{a}_{j,5} = -i\omega\lambda(\lambda + \alpha|\xi|^2)^2 \xi_{j-1} \quad (j = 2, 3, 4), \\ \tilde{a}_{55} &= (\lambda + \alpha|\xi|^2)^2 \{ \lambda^2 + (\alpha + \beta)|\xi|^2 \lambda + \gamma^2 |\xi|^2 \}, \\ \tilde{a}_{ij} &= (\lambda + \alpha|\xi|^2) \{ \lambda(\lambda + \alpha|\xi|^2)(\lambda + \kappa|\xi|^2) \delta_{ij} \\ &\quad + (\delta_{ij} |\xi|^2 - \xi_{i-1} \xi_{j-1})(\beta\lambda^2 + [\beta\kappa|\xi|^2 + \omega^2 + \gamma^2]\lambda + \gamma^2 \kappa |\xi|^2), \\ &\quad (i, j = 2, 3, 4). \end{aligned}$$

From the spectral analysis of  $\hat{A}(\xi)$  given by Matsumura and Nishida [12] (cf. Ponce [17]) we have

**LEMMA 2.1.** *Let  $\{\lambda_j(\xi)\}_{j=1}^5$  be the roots of  $\det[\lambda \cdot I + \hat{A}(\xi)] = 0$ , where  $\lambda_4(\xi) = \lambda_5(\xi) = -\alpha|\xi|^2$ . Then it follows that:*

(i)  $\lambda_j(\xi)$  depends on  $|\xi|$  only,  $\lambda_j(0) = 0$  and  $\operatorname{Re} \lambda_j(\xi) < 0$  for any  $|\xi| > 0$ ,  $j = 1, \dots, 5$ .

(ii)  $\lambda_j(\xi) \neq \lambda_k(\xi)$ ,  $j \neq k$  and  $j, k = 1, 2, 3, 4$  for all  $|\xi|$  except at most four points of  $|\xi| > 0$ .

(iii) There exist positive constants  $r_1$  such that  $\lambda_j(\xi)$  has a Taylor series expansion for  $|\xi| < r_1$  as follows:  $\lambda_1(\xi) = \overline{\lambda_2(\xi)}$  is a complex number,  $\lambda_3(\xi)$  is a real number and

$$\begin{aligned} \lambda_1(\xi) &= (\gamma^2 + \omega^2)^{1/2} (i|\xi|) + \frac{(\gamma^2 + \omega^2)(\alpha + \beta) + \omega^2 \kappa}{2(\gamma^2 + \omega^2)} (i|\xi|)^2 + \dots, \\ \lambda_3(\xi) &= \frac{\gamma^2 \kappa}{\gamma^2 + \omega^2} (i|\xi|)^2 + \frac{\gamma^2 \omega^2 \kappa^2 \{ (\gamma^2 + \omega^2)(\alpha + \beta) - \gamma^2 \kappa \}}{(\gamma^2 + \omega^2)^4} (i|\xi|)^4 + \dots. \end{aligned}$$

Similarly, there exist positive constants  $r_2 > r_1$  such that  $\lambda_j(\xi)$  has a Laurent series



expansion for  $|\xi| > r_2$  as follows: If  $\alpha + \beta \neq \kappa$ , then  $\lambda_j(\xi)$  are real numbers and

$$\begin{aligned}\lambda_1(\xi) &= (\alpha + \beta)(i|\xi|)^2 - \frac{\gamma^2\kappa - (\gamma^2 + \omega^2)(\alpha + \beta)}{(\alpha + \beta)(\alpha + \beta - \kappa)} + \dots, \\ \lambda_2(\xi) &= \kappa(i|\xi|)^2 + \frac{\omega^2}{\kappa - \alpha - \beta} + \dots, \\ \lambda_3(\xi) &= -\frac{\gamma^2}{\alpha + \beta} + \dots.\end{aligned}$$

If  $\alpha + \beta = \kappa$ , then  $\lambda_1(\xi) = \overline{\lambda_2(\xi)}$  is a complex number,  $\lambda_3(\xi)$  is a real number and

$$\begin{aligned}\lambda_1(\xi) &= \kappa(i|\xi|)^2 + \sqrt{\omega}(i|\xi|) + \dots, \\ \lambda_3(\xi) &= -\frac{\gamma^2}{\kappa} + \dots.\end{aligned}$$

(iv)  $\text{rank}[\lambda_1(\xi) \cdot I + \hat{A}(\xi)] = 3$  for all  $|\xi| > 0$  except at most one point of  $|\xi| > 0$ .

(v) The matrix exponential has the spectral resolution

$$e^{-t\hat{A}(\xi)} = \sum_{j=1}^5 e^{t\lambda_j(\xi)} P_j(\xi)$$

for all  $|\xi|$  except at most four points of  $|\xi| > 0$ .

(vi) There exists a positive constants  $\beta_0, \beta_1, \beta_2$  and  $r_1$  such that  $-\beta_0|\xi|^2 \leq \text{Re } \lambda_j(\xi) \leq -\beta_1|\xi|^2$  for  $|\xi| < r_1$  and  $\text{Re } \lambda_j(\xi) < -\beta_2$  for  $|\xi| > r_2$ ,  $j = 1, 2, \dots, 5$ .

(v)  $\|P_j(\xi)\| \leq C$  for  $|\xi| \leq r_1$ .

(vii)  $\|e^{-t\hat{A}(\xi)}\| \leq C(1+t)^3 e^{-\beta t}$  for  $|\xi| > r_1$  and a positive constant  $\beta$ .

Now we set for  $f \in X_q(R^3)$ ,  $f = {}^T\{f_j\}_{j=1}^5$

$$\begin{aligned}(2.3) \quad R_0(\lambda)f(x) &= \mathcal{F}^{-1}\{[\lambda \cdot I + \hat{A}(\xi)]^{-1}\hat{f}(\xi)\}(x) \\ &= {}^T\left\{\sum_{i=1}^5 R_{ji}(\lambda)f_i(x)\right\}_{j=1}^5,\end{aligned}$$

where  $R_{ij}(\lambda) = \mathcal{F}^{-1}\{\det[\lambda \cdot I + \hat{A}(\xi)]^{-1}\tilde{a}_{ij}(\lambda; \xi)\mathcal{F}\}$ . When  $f = {}^T\{f_1, f_2, f_5\}$  where  $f_2 = (f_2, f_3, f_4)$  we shall use the representation as follows:

$$(2.4) \quad R_0(\lambda)f(x) = {}^T\{R_{0,\rho}(\lambda)f(x), R_{0,\nu}(\lambda)f(x), R_{0,\theta}(\lambda)f(x)\}.$$

Then we shall have the following estimates of  $R_0(\lambda)f$  which is the core of our argument.

**THEOREM 2.2.** *Let  $1 < q < \infty$ ,  $b$  be a positive number and  $X_{q,b}(R^3)$  be the same symbol as in (0.7). Then for any  $f \in X_{q,b}(R^3)$  any  $\lambda \in \{\lambda \in C; \operatorname{Re} \lambda \geq 0, 0 < |\lambda| \leq 1\}$*

$$\begin{aligned} & \|R_0(\lambda)f\|_{X_q(B_b)} + \|PR_0(\lambda)f\|_{2,q,B_b} \leq C\|f\|_{X_q(R^3)}, \\ & \left\| \left(\frac{d}{d\lambda}\right)^k R_0(\lambda)f \right\|_{X_q(B_b)} + \left\| \left(\frac{d}{d\lambda}\right)^k PR_0(\lambda)f \right\|_{2,q,B_b} \\ & \leq C|\lambda|^{1/2-k}\|f\|_{X_q(R^3)}, \end{aligned}$$

where  $k$  are integers  $\geq 1$  and  $C = C(q, b, k)$  is a constant.

**PROOF.** First we note that since it follows from (2.2b), (2.2c) and Lemma 2.1 that  $F(\lambda; |\xi|) = (\lambda - \lambda_1(\xi))(\lambda - \lambda_2(\xi))(\lambda - \lambda_3(\xi))$ , we have

$$\begin{aligned} F(\lambda; |\xi|)^{-1} &= \frac{1}{\lambda_1(\xi) - \lambda_2(\xi)} \cdot \frac{1}{\lambda_1(\xi) - \lambda_3(\xi)} \cdot \frac{1}{\lambda - \lambda_1(\xi)} \\ &+ \frac{1}{\lambda_2(\xi) - \lambda_3(\xi)} \cdot \frac{1}{\lambda_2(\xi) - \lambda_1(\xi)} \cdot \frac{1}{\lambda - \lambda_2(\xi)} \\ &+ \frac{1}{\lambda_3(\xi) - \lambda_1(\xi)} \cdot \frac{1}{\lambda_3(\xi) - \lambda_2(\xi)} \cdot \frac{1}{\lambda - \lambda_3(\xi)}. \end{aligned}$$

Combining this equation and Lemma 2.1 (iii) means that

$$(2.5) \quad |F(\lambda; |\xi|)^{-1}| \leq C_\varepsilon |\lambda|^{-2\varepsilon} |\xi|^{-4+2\varepsilon} \quad \text{for } \operatorname{Re} \lambda \geq 0, \xi \in R^3 \text{ and } 0 \leq \varepsilon \leq 1,$$

and which implies that

$$(2.6) \quad |\det[\lambda + \hat{A}(\xi)]|^{-1} \leq C|\xi|^{-8} \quad \text{for } \operatorname{Re} \lambda \geq 0 \text{ and } \xi \in R^3,$$

since  $|\lambda + \alpha|\xi|^2| \geq \alpha|\xi|^2$  for  $\operatorname{Re} \lambda \geq 0$  and  $\xi \in R^3$ .

Now let  $f = {}^T\{f_j\}_{j=1}^5$ . Choosing  $\chi(r) \in C_0^\infty(R)$  so that  $\chi(r) = 1$  if  $|r| \leq 1$  and  $= 0$  if  $|r| \geq 2$ , put

$$\begin{aligned} (2.7) \quad R_{ij}(\lambda)f_j(x) &= \mathcal{F}^{-1}\{\chi(|\xi|)\det[\lambda \cdot I + \hat{A}(\xi)]^{-1}\tilde{a}_{ij}(\lambda; \xi)\hat{f}_j(\xi)\}(x) \\ &+ \mathcal{F}^{-1}\{(1 - \chi(|\xi|))\det[\lambda \cdot I + \hat{A}(\xi)]^{-1}\tilde{a}_{ij}(\lambda; \xi)\hat{f}_j(\xi)\}(x) \\ &= T_{1,ij}(\lambda)f_j(x) + T_{2,ij}(\lambda)f_j(x). \end{aligned}$$

Using Theorem 7.9.5 of [6] concerning the  $L_q$ -estimate of the Fourier multiplier,

it follows from (2.2a), (2.2d), (2.6) and (2.7) that

$$(2.8) \quad \sum_{j=1}^5 \left\| \left( \frac{d}{d\lambda} \right)^k T_{2,1j}(\lambda) f_j \right\|_{1,q,R^3} + \sum_{j=1}^5 \sum_{i=2}^5 \left\{ \left\| \left( \frac{d}{d\lambda} \right)^k T_{2,ij}(\lambda) f_j \right\|_{2,q,R^3} \right\} \\ \leq C \{ \|f_1\|_{1,q,R^3} + \|f_2\|_{q,R^3} + \|f_3\|_{q,R^3} \},$$

where  $k$  are integers  $\geq 0$  and  $C$  is a constant independent of  $|\lambda| \leq 1$ . Using a polar coordinate system, we can write as follows: for multi-index  $\alpha_i$  ( $i = 1, \dots, 5$ ):  $|\alpha_1| \leq 1, |\alpha_i| \leq 2$  ( $i = 2, \dots, 5$ )

$$(2.9) \quad \left( \frac{d}{d\lambda} \right)^k (\partial_x)^{\alpha_i} T_{1,ij}(\lambda) f_j(x) \\ = \frac{1}{(2\pi)^{3/2}} \int_{R^3} (i\xi)^{\alpha_i} e^{ix \cdot \xi} \chi(|\xi|) \left( \frac{d}{d\lambda} \right)^k \{ (\det[\lambda \cdot I + \hat{A}(\xi)])^{-1} \tilde{a}_{ij}(\lambda; \xi) \} \hat{f}_j(\xi) d\xi \\ = \frac{1}{(2\pi)^{3/2}} \int_0^2 \left( \frac{d}{d\lambda} \right)^k r^{|\alpha_i|+2} \{ (\det[\lambda \cdot I + \hat{A}(r)])^{-1} \tilde{a}_{ij}(\lambda; r\omega) \} \\ \cdot \int_{|\omega|=1} (i\omega)^{\alpha_i} e^{i(x \cdot \omega)r} \chi(r) \hat{f}_j(r\omega) dr dS_\omega,$$

where  $dS_\omega$  denote the surface element on the unit surface. By Taylor series expansion, we have

$$(2.10) \quad e^{i(x \cdot \omega)r} \chi(r) \hat{f}_j(r\omega) = \hat{f}_j(0) + \sum_{\ell=1}^{m-1} g_\ell(x, \omega) r^\ell + \int_0^1 H_m(x, \omega, s, r) ds r^m$$

where

$$g_\ell(x, \omega) = \frac{1}{\ell!} \left( \frac{\partial}{\partial r} \right)^\ell e^{i(x \cdot \omega)r} \chi(r) \hat{f}_j(r\omega) \Big|_{r=0}, \ell \geq 1, \\ H_m(x, \omega, s, r) = \frac{(1-s)^{m-1}}{(m-1)!} \left( \frac{\partial}{\partial \sigma} \right)^k e^{i(x \cdot \omega)\sigma} \chi(\sigma) \hat{f}_j(\sigma\omega) \Big|_{\sigma=sr}.$$

Note that since  $f_j \in L_{q,b}(R^3)$ , we have

$$(2.11) \quad |\hat{f}_j(0)| \leq C(b) \|f_j\|_{q,R^3}, \\ |g_\ell(x, \omega)| \leq C(b, \ell) (1 + |x|)^\ell \|f_j\|_{q,R^3}, \\ \int_0^1 |H_k(x, \omega, s, r)| ds \leq C(b, k) (1 + |x|)^k \|f_j\|_{q,R^3}.$$

In view of (2.2d), putting

$$\tilde{a}_{ij}(\lambda; r\omega) = \sum_{\beta} \tilde{a}_{\beta,ij}(\lambda; r) b_{\beta,ij}(\omega),$$

it follows from (2.9), (2.10) and (2.11) that

$$(2.12) \quad \left| \left( \frac{d}{d\lambda} \right)^k (\partial_x)^{\alpha_i} T_{1,ij}(\lambda) f_j(x) \right| \leq C(1 + |x|)^m \|f_j\|_{q, R^3} \cdot \left\{ \sum_{\beta} \sum_{\ell=0}^{m-1} \left| \int_0^1 \left( \frac{d}{d\lambda} \right)^k \{ (\det[\lambda \cdot I + \hat{A}(r)])^{-1} \tilde{a}_{\beta,ij}(\lambda; r) \} r^{|\alpha_i|+2+\ell} dr \right| + \sum_{\beta} \int_0^1 \left| \left( \frac{d}{d\lambda} \right)^k \{ (\det[\lambda \cdot I + \hat{A}(r)])^{-1} \tilde{a}_{\beta,ij}(\lambda; r) \} r^{|\alpha_i|+2+m} dr \right| \right\}.$$

In order to show that the rest of assertions in Theorem 2.2 holds, we need the following lemma.

LEMMA 2.3. *Let  $m \geq 0$ ,  $M \geq 1$  be integers. Put*

$$I_{1,m,M}(\lambda) = \int_0^1 \frac{r^m}{F(\lambda; r)^M} dr, \quad I_{2,m,M}(\lambda) = \int_0^1 \frac{r^m}{(\lambda + \alpha r^2)^M F(\lambda; r)^M} dr$$

for  $\operatorname{Re} \lambda \geq 0$ ,  $|\lambda| \leq 1$ . Then the following facts hold.

- (i)  $|I_{1,m,M}(\lambda)| \leq C(m, M)$  if  $m \geq 4M$ ,  $|I_{2,m,M}(\lambda)| \leq C(m, M)$  if  $m \geq 6M$ .
- (ii) If  $0 \leq m < 4M$ , then

$$|I_{1,m,M}(\lambda)| \leq C(m, M) \max\{|\lambda|^{m/2-2M+1/2}, |\lambda|^{m-3M+1}\} \text{ when } m \text{ is even,} \\ \leq C(m, M) \max\{|\lambda|^{m/2-2M+1/2}, |\lambda|^{m-3M+1}\} |\operatorname{Log} \lambda| \text{ when } m \text{ is odd.}$$

If  $0 \leq m < 6M$  and if  $\frac{1}{\alpha} \neq \frac{1}{\kappa} \left(1 + \frac{\omega^2}{\gamma^2}\right)$ , then

$$|I_{2,m,M}(\lambda)| \leq C(m, M) \max\{|\lambda|^{m/2-3M+1/2}, |\lambda|^{m-4M+1}\} \text{ when } m \text{ is even,} \\ \leq C(m, M) \max\{|\lambda|^{m/2-3M+1/2}, |\lambda|^{m-4M+1}\} |\operatorname{Log} \lambda| \text{ when } m \text{ is odd.}$$

- (iii) Let  $M \geq m$  and  $\ell \geq 1$  an integer. Then  $\operatorname{Re} \lambda \geq 0$ ,  $|\lambda| \leq 1$

$$\left| \int_0^1 \frac{r^{2M-2m}}{(\lambda + \alpha r^2)^{\ell} F(\lambda; r)^M} dr \right| \leq c(m, \ell, M) \max\{|\lambda|^{-M-\ell-m}, |\lambda|^{-M-\ell-2m+1}\}.$$

PROOF OF LEMMA 2.3. (i) It follows from (2.5) and the inequality  $|\lambda + \alpha r^2| \geq \alpha r^2$  when  $\operatorname{Re} \lambda \geq 0$  that (i) holds.

(ii) We shall show (ii) by using decomposition into partial fractions. We can write  $F(\lambda; r)$  as follows:

$$F(\lambda; r) = \kappa(\gamma^2 + (\alpha + \beta)\lambda)(r^2 - a_+(\lambda))(r^2 - a_-(\lambda))$$

where

$$a_{\pm}(\lambda) = -\frac{\lambda}{2\kappa} \left\{ 1 + \frac{\omega^2 + \kappa\lambda}{\gamma^2 + (\alpha + \beta)\lambda} \pm \left[ \left( 1 + \frac{\omega^2 + \kappa\lambda}{\gamma^2 + (\alpha + \beta)\lambda} \right)^2 - \frac{4\kappa\lambda}{\gamma^2 + (\alpha + \beta)\lambda} \right]^{1/2} \right\}.$$

Then we have the following estimates

$$(2.13a) \quad (a_+(\lambda) - a_-(\lambda)), \quad a_+(\lambda) = 0(\lambda) \quad \text{and} \quad a_-(\lambda) = 0(\lambda^2) \quad \text{as } \lambda \rightarrow 0,$$

$$(2.13b) \quad \left( a_+(\lambda) + \frac{\lambda}{\alpha} \right) = 0(\lambda) \quad \text{as } \lambda \rightarrow 0 \quad \text{if } \frac{1}{\alpha} \neq \frac{1}{\kappa} \left( 1 + \frac{\omega^2}{\gamma^2} \right),$$

which implies that

$$(2.14a) \quad \frac{x^m}{(x - a_+(\lambda))^M (x - a_-(\lambda))^M} = \sum_{j=1}^M \{ A_j(\lambda)(x - a_+(\lambda))^{-j} + B_j(\lambda)(x - a_-(\lambda))^{-j} \}$$

$$(2.14b) \quad |A_j(\lambda)| \leq C|\lambda|^{m-2M+j}, \quad |B_j(\lambda)| \leq C|\lambda|^{2m-3M+2j} \quad \text{for } |\lambda| \leq 1.$$

Also we have by (2.13)

$$(2.14c) \quad \frac{x^m}{\left(x + \frac{\lambda}{\alpha}\right)^M (x - a_+(\lambda))^M (x - a_-(\lambda))^M} = \sum_{j=1}^M \left\{ C_j(\lambda) \left(x + \frac{\lambda}{\alpha}\right)^{-j} + D_j(\lambda)(x - a_+(\lambda))^{-j} + E_j(\lambda)(x - a_-(\lambda))^{-j} \right\},$$

$$(2.14d) \quad |C_j(\lambda)|, \quad |D_j(\lambda)| \leq C|\lambda|^{m-3M+j} \quad \text{for } |\lambda| \leq 1 \quad \text{if } \frac{1}{\alpha} \neq \frac{1}{\kappa} \left( 1 + \frac{\omega^2}{\gamma^2} \right),$$

$$|E_j(\lambda)| \leq C|\lambda|^{2m-4M+2j} \quad \text{for } |\lambda| \leq 1 \quad \text{if } \frac{1}{\alpha} \neq \frac{1}{\kappa} \left( 1 + \frac{\omega^2}{\gamma^2} \right).$$

Moreover, putting  $a(\lambda) = -\frac{\lambda}{\alpha}$ ,  $a_{\pm}(\lambda)$ , we have by elementary calculus,

$$(2.15a) \quad \int_0^1 \frac{ds}{s - a(\lambda)} = C_1 \log|a(\lambda)| + C_2,$$

$$(2.15b) \quad \int_0^1 \frac{ds}{(s - a(\lambda))^{k+1}} = C_3 a(\lambda)^{-k} + C_4,$$

$$(2.15c) \quad \int_0^1 \frac{dr}{(r^2 - a(\lambda))^k} = C_5 a(\lambda)^{1/2-k},$$

where  $k$  are positive integers,  $C_j$  ( $j = 1, 3, 5$ ) complex constants depending only on  $k$  and  $C_j$  ( $j = 2, 4$ )  $C^\infty(\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq 0 \text{ and } |\lambda| \leq 1\})$ -functions depending also essentially on  $k$ . Combining (2.13), (2.14) and (2.15) shall reach to the statement.

(iii) Noting that

$$\begin{aligned} & \frac{r^{2M-2m}}{(\lambda + \alpha r^2)^\ell F(\lambda; r)^M} \\ &= \frac{1}{\lambda^{M+\ell+m}} \sum_{k=0}^{M+\ell+m} \binom{M+\ell+m}{k} (-\alpha)^{M+\ell+m-k} (\lambda + \alpha r^2)^{k-\ell} \cdot \frac{r^{2(2M+\ell-k)}}{F(\lambda; r)^M}, \end{aligned}$$

it follows from (ii) that

$$\begin{aligned} & \left| \int_0^1 \frac{1}{\lambda^{M+\ell+m}} \sum_{k=\ell}^{M+\ell+m} \binom{M+\ell+m}{k} (-\alpha)^{M+\ell+m-k} (\lambda + \alpha r^2)^{k-\ell} \cdot \frac{r^{2(2M+\ell-k)}}{F(\lambda; r)^M} dr \right| \\ &= \left| \sum_{k=\ell}^{M+\ell+m} \sum_{n=0}^{k-\ell} \binom{M+\ell+m}{k} \binom{k-\ell}{n} (-\alpha)^{M+\ell+m-k+n} \right. \\ & \quad \cdot \lambda^{-M-2\ell-m+k-n} \int_0^1 \frac{r^{2(2M+\ell-k+n)}}{F(\lambda; r)^M} dr \left| \right. \\ &\leq C(m, \ell, M) \max\{|\lambda|^{1/2-M-\ell-m}, |\lambda|^{-M-\ell-2m+1}\}, \end{aligned}$$

and it follows from (2.5) that

$$\begin{aligned} & \left| \int_0^1 \frac{1}{\lambda^{M+\ell+m}} \sum_{k=0}^{\ell-1} \binom{M+\ell+m}{k} (-\alpha)^{M+\ell+m-k} (\lambda + \alpha r^2)^{k-\ell} \cdot \frac{r^{2(2M+\ell-k)}}{F(\lambda; r)^M} dr \right| \\ &= \left| \frac{1}{\lambda^{M+\ell+m}} \sum_{k=0}^{\ell-1} \binom{M+\ell+m}{k} (-\alpha)^{M+\ell+m-k} \int_0^1 \frac{r^{2\ell-2k}}{(\lambda + \alpha r^2)^{\ell-k}} \cdot \frac{r^{4M}}{F(\lambda; r)^M} dr \right| \\ &\leq C(m, \ell, M) |\lambda|^{-M-\ell-m}. \end{aligned}$$

This completes the proof of Lemma 2.3.

NOW WE RETURN TO THE PROOF OF THEOREM 2.2. By direct calculation we have

$$(2.16) \quad F(\lambda; r)^k = \sum_{\ell=0}^k \sum_{n=0}^{k-\ell} \binom{k}{\ell} \binom{k-\ell}{n} \{(\alpha + \beta + \kappa)\lambda + (\gamma^2 + \omega^2)\}^\ell \cdot \{(\alpha + \beta)\kappa\lambda + \gamma^2\kappa\}^n \lambda^{3k-2\ell-3n} r^{2\ell+4n},$$

$$(2.17) \quad \left\{ \frac{d}{d\lambda} F(\lambda; r) \right\}^k = \sum_{\ell=0}^k \sum_{n=0}^{k-\ell} \binom{k}{\ell} \binom{k-\ell}{n} \{2(\alpha + \beta + \kappa)\lambda + \gamma^2 + \omega^2\}^\ell \cdot 3^{k-\ell-n} (\alpha + \beta)^n \kappa^n \lambda^{2k-2n-2\ell} r^{2\ell+4n},$$

$$(2.18a) \quad \left\{ \left( \frac{d}{d\lambda} \right)^2 F(\lambda; r) \right\}^k = \sum_{\ell=0}^k \binom{k}{\ell} 2^k 3^{k-\ell} (\alpha + \beta + \kappa)^\ell \lambda^{k-\ell} r^{2\ell},$$

$$(2.18b) \quad (\lambda + \alpha r^2)^k = \sum_{\ell=0}^k \binom{k}{\ell} \alpha^{2\ell} \lambda^{k-\ell} r^{2\ell}.$$

First when  $\frac{1}{\alpha} \neq \frac{1}{\kappa} \left( 1 + \frac{\omega^2}{\gamma^2} \right)$ , setting

$$J_1(\lambda; r) = r^4, \lambda r^2, r^2, \lambda r \text{ or } r^3,$$

$$J_2(\lambda; r) = \lambda^2 r^2, \lambda r^4, \lambda r^2 \text{ or } r^4,$$

$$G(\lambda; r) = (\lambda + \alpha r^2)F(\lambda; r),$$

it follows from (2.16), (2.17), (2.18), Appendix 1 and Lemma 2.3 that

$$(2.19) \quad \left| \int_0^1 \left( \frac{d}{d\lambda} \right)^n \{F(\lambda; r)^{-1} J_1(\lambda; r)\} r^{|\alpha_i|+2} dr \right| \\ = \left| \int_0^1 \left\{ \sum_{k=0}^2 \binom{n}{k} \left( \frac{d}{d\lambda} \right)^{n-k} F(\lambda; r)^{-1} \left( \frac{d}{d\lambda} \right)^k J_1(\lambda; r) \right\} r^{|\alpha_i|+2} dr \right| \\ \leq C \max\{1, |\lambda|^{1/2-n}\},$$

and

$$(2.20) \quad \left| \int_0^1 \left( \frac{d}{d\lambda} \right)^n \{G(\lambda; r)^{-1} J_2(\lambda; r)\} r^{|\alpha_i|+2} dr \right| \\ = \left| \int_0^1 \left\{ \sum_{k=0}^2 \binom{n}{k} \left( \frac{d}{d\lambda} \right)^{n-k} G(\lambda; r)^{-1} \left( \frac{d}{d\lambda} \right)^k J_2(\lambda; r) \right\} r^{|\alpha_i|+2} dr \right| \\ \leq C \max\{1, |\lambda|^{1/2-n}\}.$$

Also when  $\frac{1}{\alpha} = \frac{1}{\kappa} \left(1 + \frac{\omega^2}{\gamma^2}\right)$ , noting that by (2.2d) we have

$$\beta\lambda^2 + [\beta\kappa|\xi|^2 + \omega^2 + \gamma^2]\lambda + \gamma^2\kappa|\xi|^2 = \beta\lambda(\lambda + \kappa|\xi|^2) + (\omega^2 + \gamma^2)(\lambda + \alpha|\xi|^2),$$

in view of (2.19), our task is to show that

$$(2.21) \quad \left| \int_0^1 \left(\frac{d}{d\lambda}\right)^n \{G(\lambda; r)^{-1} J_3(\lambda; r)\} r^{|\alpha_i|+2} dr \right| \leq C \max\{1, |\lambda|^{1/2-n}\}$$

where  $J_3(\lambda; r) = \lambda^2 r^2$  or  $\lambda r^6$ . It follows from Lemma 2.3 (iii), (2.17), (2.18a) and Appendix 1 that (2.21) holds. Hence it follows from (2.2), (2.13), (2.19), (2.20) and (2.21) that

$$\begin{aligned} & \sum_{j=1}^5 \left\| \left(\frac{d}{d\lambda}\right)^k T_{1,ij}(\lambda) f_j \right\|_{1,q,B_b} + \sum_{j=1}^5 \sum_{i=2}^5 \left\{ \left\| \left(\frac{d}{d\lambda}\right)^k T_{1,ij}(\lambda) f_j \right\|_{2,q,B_b} \right\} \\ & \leq C \max\{1, |\lambda|^{1/2-k}\} \cdot \{\|f_1\|_{1,q,R^3} + \|f_2\|_{q,R^3} + \|f_3\|_{q,R^3}\}, \end{aligned}$$

where  $k$  are integers  $\geq 0$  and  $C$  is a constant independent of  $|\lambda| \leq 1$  and  $\operatorname{Re} \lambda \geq 0$ , and combining this with (2.8) implies that the statement of this theorem holds.

Finally in this section, we shall investigate the continuity as  $\lambda \rightarrow 0$  for the operator  $R_0(\lambda)$  and the properties for  $R_0(0)$ .

**LEMMA 2.4.** *Let  $1 < q < \infty$ ,  $b$  be a positive number and let  $f \in X_{q,b}(R^3)$ . Then  ${}^T R_0(0)f \in W_{q,\text{loc}}^1(R^3) \times W_{q,\text{loc}}^2(R^3) \times W_{q,\text{loc}}^2(R^3)$  and*

$$(2.22) \quad \lim_{R \rightarrow \infty} R^{-3} \int_{R < |x| < 2R} |R_0(0)f(x)|^q dx = 0.$$

Moreover, for any  $a > 0$  and  $0 < \varepsilon < 1/2$  the following estimates are valid:

$$(2.23) \quad \begin{aligned} & \|{}^T R_0(\lambda)f - {}^T R_0(0)f\|_{W_q^1(B_a) \times W_q^2(B_a) \times W_q^2(B_a)} \\ & \leq C(q, a, b, \varepsilon) |\lambda|^\varepsilon \|f\|_{X_q(R^3)} \end{aligned}$$

for  $\operatorname{Re} \lambda \geq 0$ ,  $|\lambda| \leq 1$  and  $f \in X_{q,b}(R^3)$ , where  $C(q, a, b, \varepsilon)$  is a constant independent of  $\operatorname{Re} \lambda \geq 0$ ,  $|\lambda| \leq 1$  and  $f \in X_{q,b}(R^3)$ .

**PROOF.** Noting that when  $\lambda = 0$

$$\hat{A}(\xi)^{-1} = \frac{1}{\gamma^2 \kappa |\xi|^4} \begin{pmatrix} \{\omega^2 + (\alpha + \beta)\kappa|\xi|^2\}|\xi|^2 & -i\gamma\kappa|\xi|^2\xi_k & -\gamma\omega|\xi|^2 \\ -i\gamma\kappa|\xi|^2\xi_j & \{\delta_{jk}|\xi|^2 - \xi_j\xi_k\}\alpha^{-1}\gamma^2\kappa & 0 \\ -\gamma\omega|\xi|^2 & 0 & \gamma^2|\xi|^2 \end{pmatrix},$$



since the kernels of Fourier integral operators in  $R_0(0)$  are the same as those of the Stokes system and the system  $\Delta$ , we have (2.22) by Lemma 2.2 and 2.3 in Iwashita [9]. Hence our task is to show (2.23). Choosing  $\chi(r) \in C_0^\infty(R)$  so that  $\chi(r) = 1$  if  $|r| \leq 1$  and  $= 0$  if  $|r| \geq 2$ , using the notations defined in (2.3) and (2.4), we have

$$\begin{aligned}
 (2.24) \quad & R_{ij}(\lambda)f_j(x) - R_{ij}(0)f_j(x) \\
 &= \mathcal{F}^{-1} \left\{ \chi(|\xi|) \left\{ \frac{\tilde{a}_{ij}(\lambda; \xi)}{\det[\lambda \cdot I + \hat{A}(\xi)]} - \frac{\tilde{a}_{ij}(0; \xi)}{\det \hat{A}(\xi)} \right\} \hat{f}_j(\xi) \right\} (x) \\
 &\quad + \mathcal{F}^{-1} \left\{ (1 - \chi(|\xi|)) \left\{ \frac{\tilde{a}_{ij}(\lambda; \xi)}{\det[\lambda \cdot I + \hat{A}(\xi)]} - \frac{\tilde{a}_{ij}(0; \xi)}{\det \hat{A}(\xi)} \right\} \hat{f}_j(\xi) \right\} (x) \\
 &= \{T_{1,ij}(\lambda) - T_{1,ij}(0)\}f_j(x) + \{T_{2,ij}(\lambda) - T_{2,ij}(0)\}f_j(x).
 \end{aligned}$$

Since it follows from (2.2a), (2.5) and (2.6) that

$$\begin{aligned}
 & \left| \xi^\eta \partial_\xi^\eta \left[ \{1 - \chi(|\xi|)\} \left\{ \frac{\tilde{a}_{11}(\lambda; \xi)}{\det[\lambda \cdot I + \hat{A}(\xi)]} - \frac{\tilde{a}_{11}(0; \xi)}{\det \hat{A}(\xi)} \right\} \right] \right| \leq C|\lambda|, \\
 & \left| \xi^\eta \partial_\xi^\eta \left[ \{1 - \chi(|\xi|)\} \left\{ \frac{\tilde{a}_{1j}(\lambda; \xi)}{\det[\lambda \cdot I + \hat{A}(\xi)]} - \frac{\tilde{a}_{1j}(0; \xi)}{\det \hat{A}(\xi)} \right\} \right] \right| \leq C \frac{|\lambda|}{|\xi|} \quad (j = 2, \dots, 5),
 \end{aligned}$$

and

$$\left| \xi^\eta \partial_\xi^\eta \left[ \{1 - \chi(|\xi|)\} \left\{ \frac{\tilde{a}_{ij}(\lambda; \xi)}{\det[\lambda \cdot I + \hat{A}(\xi)]} - \frac{\tilde{a}_{ij}(0; \xi)}{\det \hat{A}(\xi)} \right\} \right] \right| \leq C \frac{|\lambda|}{|\xi|^2} \quad (i \neq 1, j \neq 1),$$

for  $|\eta| \leq 2$ ,  $\operatorname{Re} \lambda \geq 0$ ,  $|\lambda| \leq 1$  and  $\xi \in R^3$ , by using Theorem 7.9.5 of [6] concerning the  $L_q$ -estimate of Fourier multiplier we obtain that

$$\begin{aligned}
 (2.25) \quad & \sum_{j=1}^5 \|\{T_{2,1j}(\lambda) - T_{2,1j}(0)\}f_j\|_{W_q^1(R^3)} + \sum_{j=1}^5 \sum_{i=2}^5 \|\{T_{2,ij}(\lambda) - T_{2,ij}(0)\}f_j\|_{W_q^2(R^3)} \\
 & \leq C|\lambda| \|f\|_{X_q(R^3)}.
 \end{aligned}$$

Also since it follows from (2.2a), (2.5) and (2.6) that

$$\left| \chi(|\xi|) \left\{ \frac{a_{ij}(\lambda; \xi)}{\det[\lambda \cdot I + \hat{A}(\xi)]} - \frac{a_{ij}(0; \xi)}{\det \hat{A}(\xi)} \right\} \right| \leq C|\lambda|^\varepsilon |\xi|^{-2-2\varepsilon}$$

for  $0 < \varepsilon < \frac{1}{2}$ ,  $\operatorname{Re} \lambda \geq 0$ ,  $|\lambda| \leq 1$  and  $\xi \in R^3$ , we obtain that for  $|\alpha_1| \leq 1$ ,

$$|\alpha_i| \leq 2 \quad (i \neq 1)$$

$$(2.26) \quad |\partial_x^{\alpha_i} \{T_{1,ij}(\lambda) - T_{1,ij}(0)\} f_j(x)| \\ \leq C(q, b) \|\chi(|\xi|)(i\xi)^{\alpha_i} \left\{ \frac{a_{ij}(\lambda; \xi)}{\det[\lambda \cdot I + \hat{A}(\xi)]} - \frac{a_{ij}(0; \xi)}{\det \hat{A}(\xi)} \right\}\|_{L_1(R^3)} \|f\|_{L_q(R^3)} \\ \leq C(q, b) |\lambda|^\varepsilon \|f\|_{X_{q,b}(R^3)} \quad \text{for } f \in X_{q,b}(R^3).$$

Thus it follows from (2.25), (2.26) and (2.24) that (2.23). This completes the proof.

### §3. The resolvent set of $-A$

In this section, we shall prove Theorem A. To prove this theorem we need the following lemma concerning the uniqueness, which is a key in our argument. First note that by Lemma 2.1 (iii)

$$\det[\lambda + \hat{A}(\xi)] \neq 0 \quad \text{for } \lambda \in \Sigma'' = \{\lambda \in \mathbb{C}; C_1 \operatorname{Re} \lambda + (\operatorname{Im} \lambda)^2 > 0\}$$

where  $C_1$  is a constant depending only on  $\alpha, \beta, \gamma, \kappa$ , and  $\omega$ . In the view of this and Theorem 1.5, taking a constant  $C$  in the parabolic region

$$\Sigma = \{\lambda \in \mathbb{C}; C \operatorname{Re} \lambda + (\operatorname{Im} \lambda)^2 > 0\}$$

so that  $\Sigma \subset \Sigma' \cap \Sigma''$ , we have

LEMMA 3.1. *Let  $1 < q < \infty$ . If  $\lambda \in \Sigma$ , then*

$$\operatorname{Ker}(\lambda + A) = \{0\}.$$

PROOF. Let  $(\lambda + A)u = 0$ . In view of the proof of Lemma 1.8, by bootstrap argument, we see that  ${}^T u \in W_q^{\ell+1}(\Omega) \times W_q^{\ell+2}(\Omega) \times W_q^{\ell+2}(\Omega)$  for any integer  $\ell \geq 1$ . We fix an integer  $\ell$  such that  $\ell = 0$  when  $2 \leq q < \infty$  and  $\ell \geq 3(1/q - 1/2)$  when  $1 < q < 2$ . Let  ${}^T v \in W_q^{\ell+1}(R^3) \times W_q^{\ell+2}(R^3) \times W_q^{\ell+2}(R^3)$  be functions such that  $v = u$  in  $\Omega$ . Put  $f = (\lambda + A)v$ , then since  $(\lambda + A)u = 0$  in  $\Omega$ , we see that  $\operatorname{supp} f$  is compact, and moreover  $f \in X_q^{l+1}(R^3)$ . Since  $\operatorname{supp} f$  is compact,  $f \in X_2^1(\Omega)$  when  $2 \leq q < \infty$ . When  $1 < q < 2$ , since  $\ell \geq 3(1/q - 1/2)$ , by Sobolev's imbedding theorem we have  $f \in X_2^1(\Omega)$  too. Put  $w = R_0(\lambda)f$  where the symbols are the same as in (2.4). Since  $\det[\lambda + \hat{A}(\xi)] \neq 0$  for any  $\xi \in R^3$  and  $\lambda \in \Sigma$ , by Parseval's formula we know that  ${}^T w \in W_2^1(R^3) \times W_2^2(R^3) \times W_2^2(R^3)$ . Since  $(\lambda + A)\{v - R_0(\lambda)f\} = 0$  in  $R^3$ , by Fourier transform we have  $\{\lambda + \hat{A}(\xi)\}$

$\{v(\xi) - \hat{w}(\xi)\} = 0$ , which implies that  $v = w$  in  $R^3$  because  $\det[\lambda + A(\xi)] \neq 0$ . Thus employing the same argument as in the proof of Lemma 1.8, we have  $u = 0$ . This completes the proof.

**A PROOF OF THEOREM A.** In view of Lemma 1.6, we only show (0.4). Now we shall construct parametrix to (1.1) in  $\Omega$ . Let  $\partial\Omega \subset B_{R_0}$ ,  $b$  be a fixed constant  $b > R_0 + 3$  and let  $\Omega_b = \Omega \cap B_b$ . Given  $\lambda \in \Sigma$  and  $g \in X_q(\Omega_b)$ , let  $w \in W_q^1(\Omega_b) \times W_q^2(\Omega_b) \times W_q^2(\Omega_b)$  be solutions to the problem:

$$\begin{aligned}(\lambda + A)w &= g \text{ in } \Omega_b, \\ Pw &= 0 \text{ on } \partial\Omega_b.\end{aligned}$$

The existence of such  $w$  is guaranteed by Remark 1.9. In terms of  $w$ , let us define the operator  $L(\lambda)$  by relations:

$$\begin{aligned}(3.1) \quad w &= L(\lambda)g \\ &= \{L_\rho(\lambda)g, L_\nu(\lambda)g, L_\theta(\lambda)g\}.\end{aligned}$$

Here and hereafter, for  $f \in X_q(\Omega)$ , we put  $f_0(x) = f(x)$  for  $x \in \Omega$  and  $= 0$  for  $x \in R^3 \setminus \Omega$ ,  $\Pi_b f$  stands for the restriction of  $f$  to  $\Omega_b$ . By Remark 1.9 and (3.1) we have

$$\begin{aligned}(3.2) \quad & \|L(\lambda)\Pi_b f\|_{X_q(\Omega_b)} + \|PL(\lambda)\Pi_b f\|_{2,q,\Omega_b} \\ & \leq C(q, b, \lambda) \|f\|_{X_q(\Omega)} \quad \text{for any } f \in X_q(\Omega).\end{aligned}$$

Let  $R_0(\lambda)$ ,  $R_{0,\rho}(\lambda)$ ,  $R_{0,\nu}(\lambda)$  and  $R_{0,\theta}(\lambda)$  be the same symbol as in (2.3) and (2.4). Since  $\det[\lambda + \hat{A}(\xi)] \neq 0$  whenever  $\xi \in R^3$  and  $\lambda \in \Sigma$ , by Theorem 7.9.5 of [6], we see that

$$\begin{aligned}(3.3) \quad & \|R_0(\lambda)f_0\|_{X_q(R^3)} + \|PR_0(\lambda)f_0\|_{2,q,R^3} \\ & \leq C(q, \lambda) \|f\|_{X_q(\Omega)} \quad \text{for any } f \in X_q(\Omega).\end{aligned}$$

Let  $\varphi \in C^\infty(R^3)$  such that  $\varphi(x) = 0$  for  $|x| \leq b - 2$  and  $= 1$  for  $|x| \geq b - 1$ . We introduce the operator  $\mathcal{Q}_1(\lambda)$  by the relations:

$$\begin{aligned}(3.4) \quad \mathcal{Q}_1(\lambda)f &= {}^T\{\mathcal{Q}_{1,\rho}(\lambda)f, \mathcal{Q}_{1,\nu}(\lambda)f, \mathcal{Q}_{1,\theta}(\lambda)f\} \\ &:= \varphi R_0(\lambda)(f_0) + (1 - \varphi)L(\lambda)\Pi_b f \quad \text{for any } f \in X_q(\Omega),\end{aligned}$$

Then by (3.2) and (3.3) we have

$$(3.5) \quad {}^T\mathcal{Q}_1(\lambda)f \in W_q^1(\Omega) \times W_q^2(\Omega) \times W_q^2(\Omega) \quad \text{for any } f \in X_q(\Omega),$$

$$(3.6) \quad \|\mathcal{Q}_1(\lambda)f\|_{X_q(\Omega)} + \|P\mathcal{Q}_1(\lambda)f\|_{2,q,\Omega} \\ \leq C(q, \lambda, b)\|f\|_{X_q(\Omega)} \quad \text{for any } f \in X_q(\Omega),$$

and

$$(3.7a) \quad (\lambda + A)\mathcal{Q}_1(\lambda)f = f + V(\lambda)f \text{ in } \Omega,$$

$$(3.7b) \quad P\mathcal{Q}_1(\lambda)f = 0 \text{ on } \partial\Omega,$$

where  $V(\lambda)f = {}^T\{V_\rho(\lambda)f, V_v(\lambda)f, V_\theta(\lambda)f\}$  and

$$(3.8a) \quad V_\rho(\lambda)f = \gamma\nabla\varphi[R_{0,v}(\lambda)(f_0) - L_v(\lambda)\Pi_b f],$$

$$(3.8b) \quad V_v(\lambda)f = -\alpha[\Delta\varphi + 2(\partial_j\varphi)\partial_j][R_{0,v}(\lambda)(f_0) - L_v(\lambda)\Pi_b f] \\ - \beta\nabla\{\partial_j\varphi[R_{0,v}(\lambda)(f_0) - L_v(\lambda)\Pi_b f]\}_j \\ - \beta\nabla\varphi\{\operatorname{div}[R_{0,v}(\lambda)(f_0) - L_v(\lambda)\Pi_b f]\} \\ + \gamma\nabla\varphi[R_{0,\rho}(\lambda)(f_0) - L_\rho(\lambda)\Pi_b f] \\ + \omega\partial_j\varphi[R_{0,\theta}(\lambda)(f_0) - L_\theta(\lambda)\Pi_b f]_j, \\ (3.8c) \quad V_\theta(\lambda)f = -\kappa[\Delta\varphi + 2\partial_j\varphi\partial_j][R_{0,\theta}(\lambda)(f_0) - L_\theta(\lambda)\Pi_b f] \\ + \omega\partial_j\varphi[R_{0,v}(\lambda)(f_0) - L_v(\lambda)\Pi_b f]_j.$$

Our task is to prove that  $I + V(\lambda)$  has the bounded inverse from  $X_q(\Omega)$  onto itself. It follows from (3.2), (3.3) and (3.8) that  ${}^T V(\lambda) \in \mathcal{B}(X_q(\Omega), W_q^2(\Omega) \times W_q^1(\Omega) \times W_q^1(\Omega))$  for each  $\lambda \in \Sigma$ . Since  $\operatorname{supp} V(\lambda)f \subset D_{b-1} = \{x \in \mathbb{R}^3; b-2 < |x| < b-1\}$ , by Rellich's compactness theorem  $V(\lambda)$  is a compact operator from  $X_q(\Omega)$  onto itself. Thus by Fredholm's alternative theorem, it suffices to show that  $I + V(\lambda)$  is injective in  $X_q(\Omega)$  in order to prove that  $I + V(\lambda)$  has the bounded inverse. Let  $(I + V(\lambda))f = 0$  in  $\Omega$ ,  $f \in X_q(\Omega)$ . Then it follows from (3.5), (3.7) and Lemma 3.1 that

$$\mathcal{Q}_1(\lambda)f = 0 \text{ in } \Omega,$$

$$P\mathcal{Q}_1(\lambda)f = 0 \text{ on } \partial\Omega,$$

which together with (3.4) implies that

$$(3.9a) \quad R_0(\lambda)(f_0) = 0 \quad \text{for } |x| \geq b-1,$$

$$(3.9b) \quad L(\lambda)\Pi_b f = 0 \quad \text{for } |x| \leq b-2.$$

Put  $z = \Pi_b R_0(\lambda)(f_0) - w$  where  $w = L(\lambda)\Pi_b f$  in  $\Omega_b$  and  $= 0$  in  $R^3 \setminus \Omega$ . By (3.9b) we know that  ${}^T w \in W_q^1(B_b) \times W_q^2(B_b) \times W_q^2(B_b)$  and

$$(\lambda + A)w = \Pi_b^0 f_0 \text{ in } B_b, Pw = 0 \text{ on } |x| = b,$$

where  $\Pi_b^0 f_0$  stands for the restriction of  $f_0$  to  $B_b$ , and hence we see that

$$(\lambda + A)z = 0 \text{ in } B_b, Pz = 0 \text{ on } |x| = b,$$

which with the help of Theorem 1.5 means that  $z = 0$  in  $B_b$ . As a result, we have

$$(3.10) \quad R_0(\lambda)(f_0) = L(\lambda)\Pi_b f \quad \text{in } \Omega_b.$$

Combining (3.4) and (3.10), we see that

$$(3.11) \quad \begin{aligned} R_0(\lambda)(f_0) &= \varphi\{R_0(\lambda)(f_0) - L(\lambda)\Pi_b f\} + R_0(\lambda)(f_0) \\ &= Q_1(\lambda)f = 0 \text{ in } \Omega_b. \end{aligned}$$

It follows from (3.9) and (3.11) that  $R_0(\lambda)(f_0) = 0$  in  $\Omega$ , which together with (2.1) implies that  $f_0 = f = 0$  in  $\Omega$ . Therefore, we have proved that  $(I + V(\lambda))$  has the bounded inverse  $(I + V(\lambda))^{-1}$  from  $X_q(\Omega)$  onto itself. Given  $f \in X_q(\Omega)$ , if we put  $u = Q_1(\lambda)(I + V(\lambda))^{-1}f$ , by (3.7) and (3.6) we see that  $(\lambda + A)u = f$  in  $X_q(\Omega)$  and  $u \in \mathcal{D}(A)$ , which means that the inverse  $(\lambda + A)^{-1}$  of  $(\lambda + A)$  exists, and it is bounded, that is by (3.6)

$$\begin{aligned} &\|(\lambda + A)^{-1}f\|_{X_q(\Omega)} + \|P(\lambda + A)^{-1}f\|_{2,q,\Omega} \\ &\leq C(q, b, \lambda)\|(I + V(\lambda))^{-1}\|_{\mathcal{B}(X_q(\Omega))}\|f\|_{X_q(\Omega)} \end{aligned}$$

for any  $f \in X_q(\Omega)$ , which completes the proof.

#### §4. Behaviour of $(\lambda + A)^{-1}$ near $\lambda = 0$

In this section we shall discuss behaviour of  $(\lambda + A)^{-1}$  near  $\lambda = 0$ . Our goal of this section is to prove the following theorem.

Let  $Y_q(\Omega)$  and  $Y_{q,b}(\Omega)$  be the same symbols as in (1.6) and (0.5), respectively.

**THEOREM 4.1.** *Let  $1 < q < \infty$ ,  $b_0$  a number such that  $B_{b_0} \supset R^3 \setminus \Omega$  and let  $b > b_0$ . Put  $D_\varepsilon = \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq 0, 0 < |\lambda| \leq \varepsilon\}$ ,  $\mathcal{Y} = \mathcal{B}(Y_{q,b}(\Omega); \mathcal{D}(A))$  and  $\mathcal{A}(D_\varepsilon; \mathcal{Y})$  is the set of all  $\mathcal{Y}$ -valued holomorphic functions in  $D_\varepsilon$ . Then, there exists a positive number  $\varepsilon$  and  $\tilde{R}(\lambda) \in \mathcal{A}(D_\varepsilon; \mathcal{Y})$  such that*

$$(4.1) \quad \tilde{R}(\lambda)f = (\lambda + A)^{-1}f,$$

$$(4.2a) \quad \|\tilde{R}(\lambda)f\|_{X_q(\Omega_b)} + \|P\tilde{R}(\lambda)f\|_{2,q,\Omega_b} \leq C(q, b, \varepsilon)\|f\|_{X_q(\Omega)},$$

$$(4.2b) \quad \left\| \left( \frac{d}{d\lambda} \right)^k \tilde{R}(\lambda)f \right\|_{X_q(\Omega_b)} + \left\| \left( \frac{d}{d\lambda} \right)^k P\tilde{R}(\lambda)f \right\|_{2,q,\Omega_b} \\ \leq C(q, b, k, \varepsilon)|\lambda|^{(1/2)-k}\|f\|_{X_q(\Omega)},$$

for any  $\lambda \in D_\varepsilon$ ,  $f \in Y_{q,b}(\Omega_b)$  and  $k \geq 1$  integers.

In Theorem 4.1, in view of proof of Remark 1,9, taking  $\psi \in C_0^\infty(\Omega_b)$  such that  $\int_{\Omega_b} \psi(x) dx = 1$ , we have the following corollary:

**COROLLARY 4.2.** *Let  $1 < q < \infty$ ,  $b_0$  be a number such that  $B_{b_0} \supset R^3 \setminus \Omega$  and let  $b > b_0$ . Put  $\mathcal{X} = \mathcal{B}(X_{q,b}(\Omega); \mathcal{D}(A))$ . Then, there exists a positive number  $\varepsilon$  and  $R(\lambda) \in \mathcal{A}(D_\varepsilon; \mathcal{X})$  such that  $R(\lambda)f = (\lambda + A)^{-1}f$ ,*

$$\|R(\lambda)f\|_{X_q(\Omega_b)} + \|PR(\lambda)f\|_{2,q,\Omega_b} \leq C(q, b, \varepsilon)\{\|f\|_{X_q(\Omega)} + |\lambda|^{-1}\|f_1\|_{q,\Omega}\},$$

and

$$\left\| \left( \frac{d}{d\lambda} \right)^k R(\lambda)f \right\|_{X_q(\Omega_b)} + \left\| \left( \frac{d}{d\lambda} \right)^k PR(\lambda)f \right\|_{2,q,\Omega_b} \\ \leq C(q, b, k, \varepsilon)|\lambda|^{(1/2)-k}\{\|f\|_{X_q(\Omega)} + |\lambda|^{-1}\|f_1\|_{q,\Omega}\},$$

for any  $\lambda \in D_\varepsilon$ ,  $f = {}^T\{f_1, f_2, f_3\} \in X_{q,b}(\Omega)$  and  $k \geq 1$  integers. Moreover,

$$R(\lambda) = \tilde{R}(\lambda)N_1 + \frac{\gamma}{\lambda} \tilde{R}(\lambda)N_2 + \frac{1}{\lambda} N_3$$

where  $N_j = N_j(\psi, \Omega_b)$  ( $j = 1, 2, 3$ ), are the same symbols as in (1.21).

To prove Theorem 4.1, in the same way to the proof of Theorem A we shall construct a parametrix near  $\lambda = 0$ . The following proposition concerning the uniqueness is a key in our argument, which was proved by Iwashita [9].

For an integer  $m \geq 0$  and real numbers  $\tau, q$  with  $1 < q < \infty$ , we set

$$W_q^{m,\tau}(\Omega) = \{u; (1 + |x|^2)^{\tau/2} \partial_x^\alpha u \in L_q(\Omega), |\alpha| \leq m\},$$

$$\hat{W}_q^m(\Omega) = \text{the completion of } C_0^\infty(\bar{\Omega}) \text{ by } \sum_{|\alpha|=m} \|\partial_x^\alpha \cdot\|_{q,\Omega}.$$

**PROPOSITION 4.3.** *Let  $1 < q < \infty$ . Suppose that  $u \in \hat{W}_q^2(\Omega) \cap W_q^{1,\tau}(\Omega)$  and  $p \in \hat{W}_q^1(\Omega) \cap L_q^{\tau'}(\Omega)$  with some  $\tau, \tau' \in R$  satisfy*

$$-\Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 \text{ in } \Omega,$$

$$u|_{\partial\Omega} = 0 \text{ on } \partial\Omega,$$

and

$$\lim_{R \rightarrow \infty} \frac{1}{R^3} \int_{R < |x| < 2R} |u(x)|^q dx = \lim_{R \rightarrow \infty} \frac{1}{R^3} \int_{R < |x| < 2R} |p(x)|^q dx = 0.$$

Then,  $u = 0$  and  $p = 0$  in  $\Omega$ .

**REMARK 4.4.** In view of proof of Proposition 4.3, we can replace  $\hat{W}_q^2(\Omega) \cap W_q^{1,\tau}(\Omega)$  by  $W_{q,E}^2(\Omega)$ ,  $\hat{W}_q^1(\Omega) \cap L_q^\tau(\Omega)$  by  $W_{q,E}^1(\Omega)$ , where

$$W_{q,E}^m(\Omega) = \{u; \text{there exists a } U \in W_{q,\text{loc}}^m(R^3) \text{ such that } u = U \text{ in } \Omega\}.$$

Moreover, we can show the same uniqueness theorem for the system

$$-\Delta u = 0 \text{ in } \Omega, u|_{\partial\Omega} = 0 \text{ on } \partial\Omega,$$

as Proposition 4.3.

Now we shall show the following results on uniqueness for (1.1).

**LEMMA 4.5.** *Let  $1 < q < \infty$ . Suppose that  ${}^T\{\rho, v, \theta\} \in W_{q,E}^1(\Omega) \times W_{q,E}^2(\Omega) \times W_{q,E}^2(\Omega)$  satisfies the homogeneous equation:*

$$\begin{aligned} (4.3) \quad & \gamma \operatorname{div} v = 0, \\ & -\alpha \Delta v - \beta \nabla \operatorname{div} v + \gamma \nabla \rho + \omega \nabla \theta = 0 \text{ in } \Omega, \\ & -\kappa \Delta \theta + \omega \operatorname{div} v = 0, \\ & v|_{\partial\Omega} = 0, \theta|_{\partial\Omega} = 0 \text{ on } \partial\Omega, \end{aligned}$$

and satisfies

$$(4.4) \quad \begin{aligned} \lim_{R \rightarrow \infty} \frac{1}{R^3} \int_{R < |x| < 2R} |\rho(x)|^q dx &= 0, \\ \lim_{R \rightarrow \infty} \frac{1}{R^3} \int_{R < |x| < 2R} |v(x)|^q dx &= 0, \\ \lim_{R \rightarrow \infty} \frac{1}{R^3} \int_{R < |x| < 2R} |\theta(x)|^q dx &= 0. \end{aligned}$$

Then  $\rho = 0$ ,  $v = 0$  and  $\theta = 0$  in  $\Omega$ .

PROOF. By (4.3), we have

$$(4.5) \quad -\kappa \Delta \theta = 0 \text{ in } \Omega, \theta|_{\partial\Omega} = 0 \text{ on } \partial\Omega,$$

and

$$(4.6) \quad \begin{aligned} -\alpha \Delta v + \gamma \nabla \rho &= \omega \nabla \theta \text{ in } \Omega, \\ \operatorname{div} v &= 0 \text{ in } \Omega, v|_{\partial\Omega} = 0 \text{ on } \partial\Omega. \end{aligned}$$

In view of Remark 4.4, applying Proposition 4.3 to the system (4.5) with (4.4), we have  $\theta = 0$  in  $\Omega$ , which implies  $\rho = 0$  and  $v = 0$  in  $\Omega$  by applying Proposition 4.3 to the system (4.6) with (4.4). This completes the proof.

A PROOF OF THEOREM 4.1. To prove Theorem 4.1, we shall use the symbols in the proof of Theorem A. For any  $g \in Y_{q,b}(\Omega)$ ,  $w = L(0)g$  satisfies the following relations:

$$(4.7a) \quad Aw = g \text{ in } \Omega_b, Pw = 0 \text{ on } \partial\Omega_b.$$

$$(4.7b) \quad \|w\|_{Y_q(\Omega_b)} + \|Pw\|_{2,q,\Omega_b} \leq C(q, b) \|g\|_{Y_q(\Omega_b)}.$$

Choosing  $\varphi$  in  $C^\infty(R^3)$  so that  $\varphi(x) = 1$  for  $|x| \geq b - 1$  and  $= 0$  if  $|x| \leq b - 2$ , we define the operator  $R_1(\lambda)$  by the relations:

$$(4.8a) \quad \begin{aligned} R_1(\lambda)f &= {}^T\{R_{1,\rho}(\lambda)f, R_{1,v}(\lambda)f, R_{1,\theta}(\lambda)f\} \\ &= \varphi R_0(\lambda)(f_0) + (1 - \varphi)L(0)f, \end{aligned}$$

for  $f \in Y_{q,b}(\Omega)$  and  $\lambda \in D_\varepsilon \cup \{0\}$ . Here, note that  ${}^T\{\rho, v, \theta\} = L(0)f$  satisfies the equations (1.11) and (1.13), and which implies that  $\rho = L_\rho(0)f$  is unique up to an



additive constant by Proposition 1.1. Hence,  $L_\rho(0)$  is chosen in such a way that

$$(4.8b) \quad \int_{\Omega_b} (1 - \varphi) L_\rho(0) f \, dx = \int_{B_b} R_{0,\rho}(0) f_0 \, dx - \int_{\Omega_b} \varphi R_{0,\rho}(0) f_0 \, dx.$$

Then by (4.7b), Theorem 2.2 and Lemma 2.4 we have

$$(4.9a) \quad \mathbf{R}_1(\lambda) \in \mathcal{A}(D_\varepsilon; \mathcal{Y}),$$

$$(4.9b) \quad {}^T \mathbf{R}_1(0) \in \mathcal{B}(Y_{q,b}(\Omega), W_{q,E}^1(\Omega) \times W_{q,E}^2(\Omega) \times W_{q,E}^2(\Omega)),$$

$$(4.9c) \quad (\lambda + A) \mathbf{R}_1(\lambda) \mathbf{f} = \mathbf{f} + \mathbf{S}_1(\lambda) \mathbf{f} \text{ in } \Omega, \mathbf{P} \mathbf{R}_1(\lambda) \mathbf{f} = 0 \text{ on } \partial\Omega,$$

where

$$(4.10a) \quad \mathbf{S}_1(\lambda) \mathbf{f} = {}^T \{S_{1,\rho}(\lambda) \mathbf{f}, S_{1,v}(\lambda) \mathbf{f}, S_{1,\theta}(\lambda) \mathbf{f}\},$$

and

$$(4.10b) \quad S_{1,\rho}(\lambda) \mathbf{f} = \lambda(1 - \varphi) L_\rho(0) \mathbf{f} + \gamma \nabla \varphi [R_{0,v}(\lambda)(f_0) - L_v(0) \mathbf{f}],$$

$$(4.10c) \quad \begin{aligned} S_{1,v}(\lambda) \mathbf{f} = & \lambda(1 - \varphi) L_v(0) \mathbf{f} \\ & - \alpha [\Delta \varphi + 2(\partial_j \varphi) \partial_j] [R_{0,v}(\lambda)(f_0) - L_v(0) \mathbf{f}] \\ & - \beta \nabla \{ \partial_j \varphi [R_{0,v}(\lambda)(f_0) - L_v(0) \mathbf{f}]_j \} \\ & - \beta \nabla \varphi \{ \operatorname{div} [R_{0,v}(\lambda)(f_0) - L_v(0) \mathbf{f}] \} \\ & + \gamma \nabla \varphi [R_{0,\rho}(\lambda)(f_0) - L_\rho(0) \mathbf{f}] \\ & + \omega \partial_j \varphi [R_{0,\theta}(\lambda)(f_0) - L_\theta(0) \mathbf{f}]_j, \end{aligned}$$

$$(4.10d) \quad \begin{aligned} S_{1,\theta}(\lambda) \mathbf{f} = & \lambda(1 - \varphi) L_\theta(0) \mathbf{f} \\ & - \kappa [\Delta \varphi + 2\partial_j \varphi \partial_j] [R_{0,\theta}(\lambda)(f_0) - L_\theta(0) \mathbf{f}] \\ & + \omega \partial_j \varphi [R_{0,v}(\lambda)(f_0) - L_v(0) \mathbf{f}]_j. \end{aligned}$$

It follows from (4.10), (4.9b), Theorem 2.2 and Lemma 2.4 that

$$(4.11a) \quad {}^T \mathbf{S}_1(\lambda) \in \mathcal{B}(Y_{q,b}(\Omega), W_q^1(\Omega) \times W_q^1(\Omega) \times W_q^1(\Omega)) \text{ for any } \lambda \in D_\varepsilon,$$

$$(4.11b) \quad \mathbf{S}_1(0) \in \mathcal{B}(Y_{q,b}(\Omega), X_q^1(\Omega)).$$

Noting that the Stokes formula implies that

$$\begin{aligned}
 (4.12) \quad & \int_{\Omega_b} S_{1,\rho}(\lambda) f \, dx \\
 &= \lambda \int_{\Omega_b} (1 - \varphi) L_\rho(0) f \, dx + \int_{B_b} \gamma \operatorname{div} R_{0,v}(\lambda) f_0 \, dx \\
 &\quad - \int_{\Omega_b} \varphi \gamma \operatorname{div} [R_{0,v}(\lambda) f_0 - L_v(0) f] \, dx \\
 &= \lambda \left\{ \int_{\Omega_b} (1 - \varphi) L_\rho(0) f \, dx - \int_{B_b} R_{0,\rho}(\lambda) f_0 \, dx + \int_{\Omega_b} \varphi R_{0,\rho}(\lambda) f_0 \, dx \right\},
 \end{aligned}$$

we have to modify  $S_1(\lambda)$  such that total integral over  $\Omega_b$  is zero because  $S_1(\lambda)f$  does not belong to  $Y_{q,b}(\Omega)$  when  $\lambda \neq 0$ . To do this, choosing  $\psi \in C_0^\infty(\Omega_b)$  so that  $\int_{\Omega_b} \psi(x) \, dx = 1$  and set

$$(4.13a) \quad R_2(0) = R_1(0),$$

$$(4.13b) \quad R_2(\lambda)f = {}^T\{R_{2,\rho}(\lambda)f, R_{2,v}(\lambda)f, R_{2,\theta}(\lambda)f\} \quad \text{for } \lambda \in D_\varepsilon,$$

where  $R_{2,v}(\lambda) = R_{1,v}(\lambda)$ ,  $R_{2,\theta}(\lambda) = R_{1,\theta}(\lambda)$  and

$$(4.13c) \quad R_{2,\rho}(\lambda)f = R_{1,\rho}(\lambda)f - \frac{1}{\lambda} \int_{\Omega_b} S_{1,\rho}(\lambda)f \, dx \, \psi.$$

Also, put

$$(4.14a) \quad S_2(0) = S_1(0),$$

$$(4.14b) \quad S_2(\lambda)f = {}^T\{S_{2,\rho}(\lambda)f, S_{2,v}(\lambda)f, S_{2,\theta}(\lambda)f\} \quad \text{for } \lambda \in D_\varepsilon,$$

where  $S_{2,\theta}(\lambda) = S_{1,\theta}(\lambda)$ ,

$$(4.14c) \quad S_{2,\rho}(\lambda)f = S_{1,\rho}(\lambda)f - \int_{\Omega_b} S_{1,\rho}(\lambda)f \, dx \, \psi,$$

and

$$(4.14d) \quad S_{2,v}(\lambda)f = S_{1,v}(\lambda)f - \frac{\gamma}{\lambda} \int_{\Omega_b} S_{1,\rho}(\lambda)f \, dx \, \nabla \psi.$$

Then, it follows from (4.9), (4.10), (4.13) and (4.14) that

$$(4.15a) \quad R_2(\lambda) \in \mathcal{A}(D_\varepsilon; \mathcal{Y}),$$

$$(4.15b) \quad (\lambda + A)R_2(\lambda)f = f + S_2(\lambda)f \text{ in } \Omega, \mathbf{P}R_2(\lambda)f = 0 \text{ on } \partial\Omega,$$

and by (4.10), (4.11) and (4.14) we have

$$(4.16a) \quad {}^T\mathcal{S}_2(\lambda) \in \mathcal{B}(Y_{q,b}(\Omega), W_q^1(\Omega) \times W_q^1(\Omega) \times W_q^1(\Omega)) \quad \text{for any } \lambda \in D_\varepsilon,$$

moreover, noting (4.8b) and (4.12), it follows from Lemma 2.4 that

$$(4.16b) \quad \int_{\Omega_b} \mathcal{S}_{2,\rho}(\lambda) f \, dx = 0 \quad \text{for } \lambda \in D_\varepsilon \cup \{0\},$$

$$(4.16c) \quad \|\mathcal{S}_2(\lambda) - \mathcal{S}_2(0)\|_{\mathcal{B}(Y_{q,b}(\Omega), Y_{q,b}(\Omega))} \leq C(q, b, \delta) |\lambda|^\delta$$

for  $\operatorname{Re} \lambda \geq 0$ ,  $|\lambda| \leq 1$ , where  $0 < \delta < 1/2$ . Then, we shall show the following Lemma.

**LEMMA 4.6.** *Let  $1 < q < \infty$ . Then,  $I + \mathcal{S}_2(0) \in \mathcal{B}(Y_{q,b}(\Omega))$  has the bounded inverse  $(I + \mathcal{S}_2(0))^{-1}$ .*

**PROOF.** Since  $\operatorname{supp} \mathcal{S}_2(0)f$  is contained in  $\Omega_b$ , it follows from (4.11b), (4.14a), (4.16b) and Rellich's compactness theorem,  $\mathcal{S}_2(0)$  is a compact operator from  $Y_{q,b}(\Omega)$  into itself. Thus, to prove this Lemma, by Fredholm's alternative theorem, it suffices to show that  $I + \mathcal{S}_2(0)$  is injective. Let  $(I + \mathcal{S}_2(0))f = 0$  in  $\Omega$ ,  $f \in Y_{q,b}(\Omega)$ . Our task is to prove that  $f = 0$ . It follows from (4.7b), (4.9b), (4.13a) and (4.15b) that  ${}^T\mathcal{R}_2(0)f \in W_{q,E}^1(\Omega) \times W_{q,E}^2(\Omega) \times W_{q,E}^2(\Omega)$  and satisfies

$$(4.17) \quad A\mathcal{R}_2(0)f = 0 \text{ in } \Omega, P\mathcal{R}_2(0)f = 0 \text{ on } \partial\Omega.$$

Since  $\mathcal{R}_2(0)f = \mathcal{R}_0(0)(f_0)$  for  $|x| \geq b-1$  it follows from Lemma 2.4 that

$$\lim_{R \rightarrow \infty} \frac{1}{R^3} \int_{R < |x| < 2R} |(\mathcal{R}_2(0)f)(x)|^q \, dx = 0.$$

Hence by (4.17) and Lemma 4.5 we have

$$(4.18) \quad \mathcal{R}_2(0)f = 0 \text{ in } \Omega,$$

and it follows from (4.8a), (4.13a) and (4.18) that

$$(4.19a) \quad \mathcal{R}_0(0)(f_0) = 0 \text{ for } |x| \geq b-1,$$

$$(4.19b) \quad L(0)f = 0 \text{ for } x \in \Omega_{b-2}.$$

Let us define  $w$  by the relations:  $w(x) = L(0)f(x)$  for  $x \in \Omega_b$  and  $= 0$  for  $x \in \mathbb{R}^3 \setminus \Omega_b$ , and then by (4.19) we see that  $z = \pi_b^0 \mathcal{R}_0(0)(f_0) - w$  possess the

following properties:  ${}^T\mathbf{z} \in W_q^1(B_b) \times W_q^2(B_b) \times W_q^2(B_b)$  and

$$A\mathbf{z} = 0 \text{ in } B_b, \mathbf{P}\mathbf{z} = 0 \text{ on } S_b,$$

where  $\pi_b^0 v$  is the restriction of  $v$  to  $B_b$ , and hence by Lemma 1.8 we know that  $\mathbf{z} = 0$  in  $\Omega_b$ , which means that

$$(4.20) \quad \mathbf{R}_0(0)(f_0) = \mathbf{L}(0)f \text{ in } \Omega_b.$$

Therefore, employing the same argument as in the proof of Theorem 3.1, by (4.19) and (4.20) we have  $\mathbf{f} = 0$ , which completes the proof of this Lemma.

We return to the proof of Theorem 4.1. In view of Lemma 4.6,  $(I + \mathbf{S}_2(0))^{-1} \in \mathcal{B}(Y_{q,b}(\Omega))$ , and then put

$$M = \|(I + \mathbf{S}_2(0))^{-1}\|,$$

where  $\|\cdot\|$  stands for the operation norm. By (4.16c) and Neumann series expansion, there exists an  $\varepsilon > 0$  such that  $I + \mathbf{S}_2(\lambda)$  also has the bounded inverse  $(I + \mathbf{S}_2(\lambda))^{-1}$  from  $Y_{q,b}(\Omega)$  onto itself whenever  $\lambda \in D_\varepsilon$ , and moreover

$$(4.21) \quad \|(I + \mathbf{S}_2(\lambda))^{-1}\| \leq 2M \quad \text{for } \lambda \in D_\varepsilon.$$

If we look at (4.13) with (4.8) and (4.10), by Theorem 2.2 we have

$$(4.22a) \quad \|\mathbf{R}_2(\lambda)\mathbf{f}\|_{X_q(\Omega_b)} + \|\mathbf{P}\mathbf{R}_2(\lambda)\|_{2,q,\Omega_b} \leq C(\varepsilon, b)\|\mathbf{f}\|_{X_q(\Omega)},$$

$$(4.22b) \quad \left\| \left( \frac{d}{d\lambda} \right)^k \mathbf{R}_2(\lambda)\mathbf{f} \right\|_{X_q(\Omega_b)} + \left\| \left( \frac{d}{d\lambda} \right)^k \mathbf{P}\mathbf{R}_2(\lambda) \right\|_{2,q,\Omega_b} \\ \leq C(\varepsilon, b)|\lambda|^{1/2-k}\|\mathbf{f}\|_{X_q(\Omega)}, \quad k \geq 1,$$

for  $\mathbf{f} \in Y_{q,b}(\Omega)$  and  $\lambda \in D_\varepsilon$ . Put

$$\tilde{\mathbf{R}}(\lambda) = \mathbf{R}_2(\lambda)(I + \mathbf{S}_2(\lambda))^{-1},$$

and then by (4.15) we see that  $\tilde{\mathbf{R}}(\lambda)\mathbf{f} \in \mathcal{D}(\mathbf{A})$  and

$$(4.23) \quad (\lambda + \mathbf{A})\tilde{\mathbf{R}}(\lambda)\mathbf{f} = \mathbf{f} \text{ in } \Omega$$

for any  $\lambda \in D_\varepsilon$  and  $\mathbf{f} \in Y_{q,b}(\Omega)$ . In particular, when  $\mathbf{f} \in Y_{q,b}(\Omega)$ , by (4.23) and Lemma 3.1 we have  $\tilde{\mathbf{R}}(\lambda)\mathbf{f} = (\lambda + \mathbf{A})^{-1}\mathbf{f}$  for  $\lambda \in D_\varepsilon$  and  $\mathbf{f} \in Y_{q,b}(\Omega)$ . Combining (4.21), (4.22) we have (4.1) and (4.2), which completes the proof of Theorem 4.1.

### §5. Proofs of Theorem B and Corollary C

In this section, we shall prove Theorem B and Corollary C. To do this we prepare the following lemma, which was proved by Shibata. (see Theorem 3.2 and 3.7 of [18])

LEMMA 5.1. *Let  $X$  be a Banach space with norm  $|\cdot|_X$ . Let  $f(\tau)$  be a function of  $C^\infty(R - \{0\}; X)$  such that  $f(\tau) = 0$ ,  $|\tau| \geq a$  with some  $a > 0$ . Assume that there exists a constant  $C(f)$  depending on  $f$  such that for any  $0 < |\tau| \leq a$ ,*

$$\left| \left( \frac{d}{d\tau} \right)^k f(\tau) \right|_X \leq C(f) |\tau|^{-1/2-k}, k = 0, 1.$$

Put  $g(t) = \int_{-\infty}^{\infty} f(\tau) e^{-i t \tau} d\tau$ . Then

$$|g(t)|_X \leq C(1+t)^{-1/2} C(f).$$

Now we shall prove Theorem B. In view of the facts that when  $0 < t \leq 1$  by Theorem A we have

$$\begin{aligned} & \|\partial_t^M e^{-tA} \mathbf{u}\|_{X_q(\Omega)} + \|\mathbf{P} \partial_t^M e^{-tA} \mathbf{u}\|_{2,q,\Omega} \\ & \leq C \|(1+A)^{M+N} e^{-tA} \mathbf{u}\|_{X_q(\Omega)} \leq C t^{-N-M} \|\mathbf{u}\|_{X_q(\Omega)} \end{aligned}$$

for any  $\mathbf{u} \in X_q(\Omega)$  and any integers  $N \geq 1, M \geq 0$ , we have only to show the case  $t \geq 1$ . Note that by Corollary 7.5 of [16, Chapter 1] we can write

$$\begin{aligned} (5.1) \quad e^{-tA} \mathbf{u} &= \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{t\lambda} (\lambda + A)^{-1} \mathbf{u} d\lambda \\ &= -\frac{1}{2\pi i t} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{t\lambda} \frac{d}{d\lambda} (\lambda + A)^{-1} \mathbf{u} d\lambda \end{aligned}$$

for all  $\mathbf{u} \in \mathcal{D}(A^2)$ , because

$$(5.2) \quad \left\| \frac{d}{d\lambda} (\lambda + A)^{-1} \mathbf{u} \right\|_{X_q(\Omega)} \leq \frac{C(\varepsilon)}{1+|\lambda|^2} \|\mathbf{u}\|_{X_q(\Omega)} \quad \text{for any } \operatorname{Re} \lambda \geq \varepsilon > 0$$

by Theorem A. Since  $\mathcal{D}(A^2)$  is dense in  $X_q(\Omega)$ , the equation (5.1) holds in  $X_q(\Omega)$ .

Let  $\mathbf{u} \in Y_{q,b}(\Omega)$ ,  $b > b_0$  and let  $\psi \in C_0^\infty(\mathbb{R}^3)$  such that  $\psi(x) = 1$  for  $|x| \leq b$  and  $= 0$  for  $|x| \geq b+1$ . Since we can move the path in the following integral to

the imaginary axis by Theorem 4.1, (5.1) and (5.2), we have

$$\begin{aligned}\partial_X^\alpha \psi e^{-tA} u &= \frac{-1}{2\pi i t} D_X^\alpha \left\{ \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{t\lambda} \psi \frac{d}{d\lambda} (\lambda + A)^{-1} u d\lambda \right\} \\ &= \frac{-1}{2\pi i t} D_X^\alpha \left\{ \int_{-\infty}^{\infty} e^{its} \psi \frac{d}{ds} (is + A)^{-1} u ds \right\}\end{aligned}$$

for any  $u \in Y_{q,b}(\Omega)$  and multi-index  $\alpha_i$  ( $i = 1, 2, 3$ ) :  $|\alpha_1| \leq 1, |\alpha_i| \leq 2$  ( $i = 2, 3$ ) where  $D_X^\alpha = {}^T\{(\partial_X)^{\alpha_1}, (\partial_X)^{\alpha_2}, (\partial_X)^{\alpha_3}\}$ . Taking  $\eta(s) \in C^\infty(\mathbb{R})$  so that  $\eta(s) = 1$  for  $|s| \leq 1/4$  and  $= 0$  for  $|s| \geq 1/2$  we have

$$(5.3) \quad D_X^\alpha \psi e^{-tA} u = J_0(t)u + J_\infty(t)u$$

where

$$\begin{aligned}J_0(t)u &= \frac{-1}{2\pi i t} D_X^\alpha (\psi \int_{-\infty}^{\infty} e^{its} \eta(s) \frac{d}{ds} (is + A)^{-1} u ds), \\ J_\infty(t)u &= \frac{-1}{2\pi i t} D_X^\alpha (\psi \int_{-\infty}^{\infty} e^{its} (1 - \eta(s)) \frac{d}{ds} (is + A)^{-1} u ds).\end{aligned}$$

By Theorem A we have

$$\begin{aligned}(5.4) \quad & \|D_X^\alpha (1 - \eta(s)) \left(\frac{d}{ds}\right)^N (is + A)^{-1} u\|_{q,\Omega} \\ & \leq (1 - \eta(s)) \{ \|(is + A)^{-N-1} u\|_{X_q(\Omega)} + \|P(is + A)^{-N-1} u\|_{2,q,\Omega} \} \\ & \leq C(N)(1 + |s|)^{-N} \|u\|_{X_q(\Omega)},\end{aligned}$$

and hence by the relation  $(1/t) \cdot (d/d\lambda) e^{t\lambda} = e^{t\lambda}$ , we have

$$(5.5) \quad \|\partial_t^M J_\infty(t)u\|_{q,\Omega} \leq C(N, M, \alpha) t^{-N} \|u\|_{X_q(\Omega)}$$

for any integers  $N \geq 2, M \geq 0$ . On the other hand, noting that

$$\begin{aligned}\partial_t^M J_0(t)f &= \frac{-1}{2\pi} \sum_{n=0}^M \binom{M}{n} \partial_t^{M-n} t^{-1} D_X^\alpha \left\{ \psi \int_{-\infty}^{\infty} e^{ist} \eta(s) (is)^n \frac{d}{ds} \tilde{R}(is) f ds \right\} \\ &= -t^{-(M+1)} \sum_{n=0}^M c(n) D_X^\alpha \left\{ \psi \int_{-\infty}^{\infty} e^{ist} \left(\frac{d}{ds}\right)^n \{ \eta(s) (is)^n \frac{d}{ds} \tilde{R}(is) f \} ds \right\},\end{aligned}$$

it follows from Theorem 4.1 and Lemma 5.1 that

$$(5.6) \quad \|\partial_t^M J_0(t)u\|_{q,\Omega} \leq C(M, b, q) (1+t)^{-(M+3/2)} \|u\|_{X_q(\Omega)}$$

for any  $u \in Y_{q,b}(\Omega)$ , integer  $M \geq 0$  and  $t \geq 1$ . Combining (5.3), (5.5) and (5.6) we have for any  $u \in Y_{q,b}(\Omega)$ , integer  $M \geq 0$  and  $t \geq 1$

$$(5.7) \quad \|\partial_t^M e^{-tA} u\|_{Y_{q,b}(\Omega)} + \|\partial_t^M P e^{-tA} u\|_{2,q,\Omega_b} \leq C(1+t)^{-3/2-M} \|u\|_{Y_{q,b}(\Omega)}.$$

This completes the proof of Theorem B.

Next we shall prove Corollary C. Let  $u \in X_{q,b}(\Omega)$ . Taking  $\phi \in C_0^\infty(\Omega_b)$ , such that  $\int_{\Omega_b} \phi(x) dx = 1$ , in view of Remark 1.9, we have

$$(\lambda + A)^{-1} u = (\lambda + A)^{-1} N_1 u + \frac{\gamma}{\lambda} (\lambda + A)^{-1} N_2 u + \frac{1}{\lambda} N_3 u \quad \text{for } u \in X_{q,b}(\Omega)$$

where  $N_j = N_j(\phi, \Omega_b)$  ( $j = 1, 2, 3$ ) be the same symbol as in (1.21). Combining this and (5.1), we have

$$(5.8) \quad \begin{aligned} e^{-tA} u &= \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{t\lambda} (\lambda + A)^{-1} N_1 u d\lambda \\ &\quad + \frac{\gamma}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{t\lambda} (\lambda + A)^{-1} N_2 u \frac{d\lambda}{\lambda} \\ &\quad + \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{1}{\lambda} e^{t\lambda} N_3 u d\lambda. \end{aligned}$$

Putting  $T_1(b, \phi, t)u = e^{-tA} N_1 u$  and  $T_2(b, \phi, t)u = \gamma \int_0^t e^{-sA} N_2 u ds + N_3 u$ , since  $\frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{1}{\lambda} e^{t\lambda} u d\lambda = u$  for any  $u \in X_q(\Omega)$ , and since by Theorem 7.4 of [16, Chapter 1] we have

$$\int_0^t e^{-sA} u ds = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{t\lambda} (\lambda + A)^{-1} u \frac{d\lambda}{\lambda} \quad \text{for } u \in \mathcal{D}(A) \text{ and } t > 0,$$

it follows from (5.1) and (5.8) that the relation (0.8) holds. Moreover, nothing that  $N_1 u, N_2 u \in Y_{q,b}(\Omega)$ , since by (5.1) and (5.8) we have

$$(5.9) \quad \begin{aligned} \partial_t e^{-tA} u &= \partial_t \left\{ \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{t\lambda} (\lambda + A)^{-1} N_1 u d\lambda \right\} \\ &\quad + \frac{-\gamma}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{t\lambda} (\lambda + A)^{-1} N_2 u d\lambda, \end{aligned}$$

it follows from (5.7), (5.8) and (5.9) that the estimates (0.9) and (0.10) hold. This completes the proof of Corollary C.

APPENDIX 1. Let  $n$  be an integer  $\geq 0$  and let

$$F(\lambda; r) = \lambda^3 + (\alpha + \beta + \kappa)r^2\lambda^2 + \{(\alpha + \beta)\kappa r^2 + \gamma^2 + \omega^2\}r^2\lambda + \gamma^2\kappa r^4.$$

Then

$$\begin{aligned} \text{(App1)} \quad \left(\frac{d}{d\lambda}\right)^n F(\lambda; r)^{-1} &= \sum_{0 \leq \ell \leq [(n-\ell)/2]} \sum_{k=0}^{[(n-\ell)/2]-\ell} C(k, \ell, n) \{F(\lambda; r)^{-n-1+2\ell+k} \\ &\quad \times \left\{\frac{d}{d\lambda} F(\lambda; r)\right\}^{n-3\ell-2k} \left\{\left(\frac{d}{d\lambda}\right)^2 F(\lambda; r)\right\}^k\}. \end{aligned}$$

Moreover, set  $G(\lambda; r) = (\lambda + \alpha r^2)F(\lambda; r)$ , then

$$\begin{aligned} \text{(App2)} \quad \left(\frac{d}{d\lambda}\right)^n G(\lambda; r)^{-1} &= \sum_{m=0}^n \sum_{0 \leq \ell \leq [(m-\ell)/2]} \sum_{k=0}^{[(m-\ell)/2]-\ell} C(m, k, \ell, n) \left\{G(\lambda; r)^{-n-1} F(\lambda; r)^{n-m+2\ell+k} \right. \\ &\quad \times \left\{\frac{d}{d\lambda} F(\lambda; r)\right\}^{m-3\ell-2k} \left\{\left(\frac{d}{d\lambda}\right)^2 F(\lambda; r)\right\}^k \left. \right\} (\lambda + \alpha r^2)^m. \end{aligned}$$

PROOF. Since it directly follows from (App1) and Leibniz rule that (App2) holds, our task is to show (App1). Now we shall show (App1) by induction on  $n$ . When  $n = 0$ , obviously (App1) holds. Assume that  $n \geq 1$  and that (App1) and that (App1) is valid for smaller values of  $n$ . Noting that  $(d/d\lambda)^3 F(\lambda; r) = 6$ , we have

(App3)

$$\begin{aligned} \frac{d}{d\lambda} I_n(\lambda; r) &= \frac{d}{d\lambda} \left\{ \sum_{k=0}^{[(n-\ell)/2]-\ell} C(k, \ell, n) F(\lambda; r)^{-n-1+2\ell+k} \left\{\frac{d}{d\lambda} F(\lambda; r)\right\}^{n-3\ell-2k} \left\{\left(\frac{d}{d\lambda}\right)^2 F(\lambda; r)\right\}^k \right\} \\ &= \sum_{k=0}^{[(n-\ell)/2]-\ell} C(k, \ell, n) F(\lambda; r)^{-n-2+2\ell+k} \left\{\frac{d}{d\lambda} F(\lambda; r)\right\}^{n+1-3\ell-2k} \left\{\left(\frac{d}{d\lambda}\right)^2 F(\lambda; r)\right\}^k \\ &\quad + \sum_{k=1}^{[(n-\ell)/2]-\ell+1} C(k, \ell, n) F(\lambda; r)^{-n-2+2\ell+k} \left\{\frac{d}{d\lambda} F(\lambda; r)\right\}^{n+1-3\ell-2k} \left\{\left(\frac{d}{d\lambda}\right)^2 F(\lambda; r)\right\}^k \\ &\quad + \sum_{k=0}^{[(n-\ell)/2]-\ell-1} C(k, \ell, n) F(\lambda; r)^{-n+2\ell+k} \left\{\frac{d}{d\lambda} F(\lambda; r)\right\}^{n-2-3\ell-2k} \left\{\left(\frac{d}{d\lambda}\right)^2 F(\lambda; r)\right\}^k \\ &= I_{n,1}(\lambda; r) + I_{n,2}(\lambda; r) + I_{n,3}(\lambda; r). \end{aligned}$$



Since  $[(n - \ell)/2] - \ell = [(n + 1 - \ell)/2] - \ell$  when both  $n$  and  $\ell$  are even or odd, and since  $[(n - \ell)/2] - \ell = [(n + 1 - \ell)/2] - \ell - 1$  when  $n$  (resp.  $\ell$ ) is even and  $\ell$  (resp.  $n$ ) is odd, we have

$$(App4) \quad I_{n,1}(\lambda; r) = I_{n+1}(\lambda; r).$$

Also since  $n - 3\ell - 2([(n - \ell)/2] - \ell) = 0$  when both  $n$  and  $\ell$  are even or odd, and since  $n - 3\ell - 2([(n - \ell)/2] - \ell) = 1$  when  $n$  (resp.  $\ell$ ) is even and  $\ell$  (resp.  $n$ ) is odd, we have

$$(App5) \quad I_{n,2}(\lambda; r) = I_{n+1}(\lambda; r).$$

Note that  $0 \leq \ell \leq m$  if  $0 \leq \ell \leq [(n - \ell)/2]$  and  $n = 3m + k$  ( $k = 0, 1, 2$ ). When  $n = 3m$ ,  $3m + 1$ , since  $0 \leq \ell \leq m$  if  $0 \leq \ell \leq [(n + 1 - \ell)/2]$ , it follows from (App3), (App4), (App5) and the induction assumption that

$$\begin{aligned} \left(\frac{d}{d\lambda}\right)^{n+1} F(\lambda; r)^{-1} &= \sum_{\ell=0}^m \frac{d}{d\lambda} I_n(\lambda; r) \\ &= \sum_{\ell=0}^m I_{n+1}(\lambda; r) + \sum_{\ell=0}^{m-1} I_{n,3}(\lambda; r) \\ &= \sum_{\ell=0}^m I_{n+1}(\lambda; r) \\ &= \sum_{0 \leq \ell \leq [(n+1-\ell)/2]} I_{n+1}(\lambda; r). \end{aligned}$$

Similarly, when  $n = 3m + 2$ , since  $0 \leq \ell \leq m + 1$  if  $0 \leq \ell \leq [(n + 1 - \ell)/2]$ , and since  $[(n - \ell)/2] - \ell - 1 = 0$  if  $\ell = m$ , it follows from (App3), (App4), (App5) and the induction assumption that

$$\begin{aligned} \left(\frac{d}{d\lambda}\right)^{n+1} F(\lambda; r)^{-1} &= \sum_{\ell=0}^m \frac{d}{d\lambda} I_n(\lambda; r) \\ &= \sum_{\ell=0}^m I_{n+1}(\lambda; r) + \sum_{\ell=0}^m I_{n,3}(\lambda; r) \\ &= \sum_{\ell=0}^m I_{n+1}(\lambda; r) + \sum_{\ell=1}^{m+1} I_{n+1}(\lambda; r) \\ &= \sum_{0 \leq \ell \leq [(n+1-\ell)/2]} I_{n+1}(\lambda; r). \end{aligned}$$

This completes the proof.

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